The h^1 error of the Galerkin approximation

Problem 1. The h^1 error of the Galerkin approximation to a function $f(x)$.

Let us consider a uniform partition of $[a, b]$ by choosing points $a = x_0 < x_1 < \cdots < x_{n+1} = b$ with $h = x_{i+1} - x_i = \frac{b-a}{n+1}$, $i = 0, 1, \ldots, n$. We define the test functions as

$$\phi_i(x) = \begin{cases} 0, \quad a \le x \le x_{i-1}, \\ \frac{1}{h}(x - x_{i-1}), \quad x_{i-1} \le x \le x_i, \\ \frac{1}{h}(x_{i+1} - x), \quad x_i \le x \le x_{i+1}, \\ 0, \quad x_{i+1} \le x \le b, \end{cases}$$

for $i = 1, 2, 3, \ldots, n$. We seek an approximation in the form $f_a(x) = \sum\limits_{j=1}^{n} c_j \phi_j(x) \approx f(x)$, where the coefficients c_j are assumed to satisfy the following equalities

$$\int\limits_a^b f_a(x)\phi_i(x)dx = \int\limits_a^b \sum_{j=1}^{n} c_j\phi_j(x)\phi_i(x)dx = \int\limits_a^b f(x)\phi_i(x)dx, \ i = 1, 2, \ldots, n$$

This equalities can rewritten in a form of the system of the linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},$$

where

$$a_{ij} = \int\limits_a^b \phi_i(x)\phi_j(x)dx, \quad f_i = \int\limits_a^b f(x)\phi_i(x)dx, \quad i.j = 1, 2, \ldots, n.$$

We calculate the above integrals numerically.

$$\int\limits_{x_i}^{x_{i+1}} g(x)dx \approx (g(x_i) + 4g(x_{i+\frac{1}{2}}) + g(x_{i+1}))\frac{h}{6}, \quad x_{i+\frac{1}{2}} = x_i +$$

$$a_{ii} = \int\limits_a^b \phi_i^2(x)dx = \int\limits_{x_{i-1}}^{x_{i+1}} \phi_i^2(x)dx \approx \frac{2}{3}h, \quad i = 1, 2, \ldots, n$$

$$a_{i,i+1} = \int\limits_{x_i}^{x_{i+1}} \phi_i(x)\phi_{i+1}(x)dx \approx \frac{h}{6}, \quad a_{i+1,i} = \int\limits_{x_i}^{x_{i+1}} \phi_i(x)\phi_{i+1}(x)dx \approx \frac{h}{6}, \quad i = 1, 2, \ldots, n-1$$

$$\int\limits_{x_i}^{x_{i+1}} g(x)dx \approx g(x_{i+\frac{1}{2}})h$$

$$f_i = \int\limits_a^b f(x)\phi_i(x)dx = \int\limits_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)dx \approx (f(x_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}}))\frac{h}{2}$$

The h^1 error of the approximation

$$er_{2;1}(h) = \left(\int_a^b (f(x) - f_a(x))^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b (f'(x) - f'_a(x))^2 dx\right)^{\frac{1}{2}} =$$

$$\left(\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (f(x) - f_a(x))^2 dx\right)^{\frac{1}{2}} + \left(\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (f'(x) - f'_a(x))^2 dx\right)^{\frac{1}{2}} =$$

$$\int_{x_i}^{x_{i+1}} (f(x) - f_a(x))^2 dx \approx ((f(x_i) - f_a(x_i))^2 + 4(f(x_{i+\frac{1}{2}}) - f_a(x_{i+\frac{1}{2}}))^2 + (f(x_{i+1}) - f_a(x_{i+1}))^2)\frac{h}{6}$$

$$\int_{x_i}^{x_{i+1}} (f'(x) - f'_a(x))^2 dx \approx ((f'(x_i) - f'_a(x_i))^2 + 4(f'(x_{i+\frac{1}{2}}) - f'_a(x_{i+\frac{1}{2}}))^2 + (f'(x_{i+1}) - f'_a(x_{i+1}))^2)\frac{h}{6}$$

For calculations let you take: $f(x) = \sin x$, $a = 0$, $b = \pi$. Observing the error

$$\mathrm{er}_{2;1}(h) = \left(\int_a^b (f(x) - f_a(x))^2 dx\right)^{1/2} + \left(\int_a^b (f'(x) - f'_a(x))^2 dx\right)^{1/2}$$

for $n = 50, 100, 200, 400, 800$ try to determine the accuracy order of the approximation method.