Numerical methods in differential equations 2 (7 marca 2024)

Solutions to the problems below should be prepared in the form of Matlab or Python codes with attached flowcharts explaining your algorithms.

Problem 1. Given the initial value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \leq t \leq 2, \quad y(1) = -1$$

with exact solution

$$y(t) = -1/t.$$

a) Use Euler's methods (explicit and implicit) with $h = 0.05$ to approximate the solution and compare it with the actual values of y.
b) Use the results obtained in part (a) and linear interpolation to approximate the following values of y and compare them to the actual values.

i) $y(1.04)$, ii) $y(1.55)$, iii) $y(1.97)$.

c) Determine numerically the value of h necessary for $\max_{0 \leq i \leq n} |y(t_i) - y_i| \leq 0.1$.

Problem 2. Obtain an approximation to the solution of

$$y' = 1 + t\sin(ty), \quad 0 \leq t \leq 2, \quad y(0) = 0$$

using the explicit Euler's method with $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$ (equivalently $n = 20, 40, 80, 160, 320$). Can the error bound be obtained using an estimate of the convergence rate of the explicit Euler's method?

Problem 3. Use the Runge-Kutta method of order two for the initial value problem

$$y' = -y + t + 1, \quad 0 \leq t \leq 1, \quad y(0) = \frac{1}{3}$$

with $h = 0.1, 0.05, 0.025, 0.0125$.
a) Find the exact solution $y(t)$.
b) Let $y_{i+1} = y_i + \Phi(t_i, y_i, h)$, where $\Phi(t_i, y_i, h)$ denotes all the transformations needed to get y_{i+1} from y_i described in the Runge-Kutta method. Consider the local truncation error $e_l^i(h) = y(t_{i+1}) - y(t_i) - \Phi(t_i, y(t_i), h)$ for chosen values of h. Are the results consistent with the differential problem?

Problem 4. Solve the following ”stiff”initial value problems using

(i) explicit Euler's method,
(ii) implicit Euler's method,
(iii) Runge-Kutta fourth-order method
(iv) the trapezoidal method.

a)
$$y' = -5y, \ \ 0 \leq t \leq 1, \ \ y(0) = e.$$

Use $h = 0.1$ and compare the result to the actual solution $y(t) = e^{1-5t}$.

b)
$$y' = -7(y - t) + 1, \ \ 0 \leq t \leq 1, \ \ y(0) = 3.$$

Use $h = 0.1$ and compare the result to the actual solution $y(t) = t + 3e^{-7t}$.

c)
$$y' = -20(y - t^2) + 2t, \ \ 0 \leq t \leq 1, \ \ y(0) = \frac{1}{3}.$$

Use $h = 0.05$ for $0 \leq t \leq 0.2$ and $h = 0.1$ for $0.2 \leq t \leq 1$ and compare the result to the actual solution $y(t) = t^2 + \frac{1}{3}e^{-20t}$.

d)
$$y' = -20y + 20\sin t + \cos t, \ \ 0 \leq t \leq 1, \ \ y(0) = 1.$$

Use $h = 0.01$ for $0 \leq t \leq 0.2$ and $h = 0.05$ for $0.2 \leq t \leq 1$ and compare the result to the actual solution $y(t) = e^{-20t} + \sin t$.

e)
$$y' = (50/y) - 50y, \ \ 0 \leq t \leq 1, \ \ y(0) = \sqrt{2}.$$

Use $h = 0.05$ for $0 \leq t \leq 0.2$ and $h = 0.1$ for $0.2 \leq t \leq 1$ and compare the result to the actual solution $y(t) = [1 + e^{-100t}]^{1/2}$.

f)
$$\begin{cases} u_1' = 32u_1 + 66u_2 + \frac{2}{3}t + \frac{2}{3}, \ \ 0 \leq t \leq 1, \ \ u_1(0) = \frac{1}{3}; \\ u_2' = -66u_1 - 133u_1 - \frac{1}{3}t - \frac{1}{3}, \ \ 0 \leq t \leq 1, u_2(0) = \frac{1}{3}. \end{cases}$$

Use $h = 0.01$ for $0 \leq t \leq 0.1$ and $h = 0.1$ for $0.1 \leq t \leq 1$ and compare the result to the actual solution

$$u_1(t) = \frac{2}{3}t + \frac{2}{3}e^{-t} - \frac{1}{3}e^{-100t}, \ \ u_2(t) = -\frac{1}{3}t - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-100t}.$$

'compare the result to the actual solution' means that you should compute $\max\limits_{1 \leq i \leq n} |y(t_i) - y_i|$ and deduce the convergence rate of the method. The following formulas are useful in an application of the implicit Euler's method given in Problem 4.

$$a)\ y_{n+1} = y_n + h(-5y_{n+1}),\quad y_{n+1} = \frac{1}{1+5h} y_n.$$

$$b)\ \ y_{n+1} = y_n + h(-7(y_{n+1} - t_{n+1}) + 1),$$
$$y_{n+1} = \frac{1}{1+7h}(y_n + h(-7(-t_{n+1}) + 1))$$

$$c)\ \ y_{n+1} = y_n + h(-20(y_{n+1} - t_{n+1}^2) + 2t_{n+1}),$$
$$y_{n+1} = \frac{1}{1+20h}(y_n + h(-20(-t_{n+1}^2) + 2t_{n+1}))$$

$$d)\ \ y_{n+1} = y_n + h(-20y_{n+1} + 20\sin t_{n+1} + \cos t_{n+1}),$$
$$y_{n+1} = \frac{1}{1+20h}(y_n + h(20\sin t_{n+1} + \cos t_{n+1}))$$

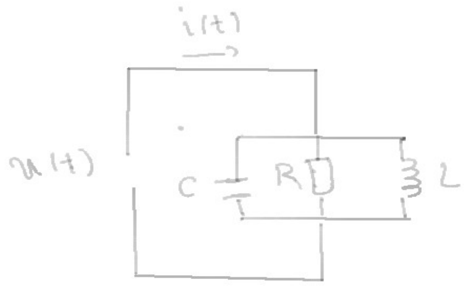
$$e)\ \ y_{n+1} = y_n + h((50/y_{n+1}) - 50y_{n+1}),$$
$$(1+50h)y_{n+1}^2 - y_n y_{n+1} - 50h = 0$$

$$y_{n+1} = \frac{y_n \pm \sqrt{y_n^2 + 4\cdot 50h\cdot(1+50h)}}{2(1+50h)}$$

$$f)\ \ u_{n+1} = u_n + h(Au_{n+1} + f(t_{n+1})),\ \ (Id - hA)u_{n+1} = u_n + hf(t_{n+1})$$

$$A = \begin{bmatrix} 32 & 66 \\ -66 & -133 \end{bmatrix},\ f(t) = \begin{bmatrix} \frac{2}{3}t + \frac{2}{3} \\ -\frac{1}{3}t - \frac{1}{3} \end{bmatrix}$$

Problem 5. In a circuit with impressed voltage U, and resistance R, inductance L, capacitance C in parallel, the current i



satisfies the differential equation

$$\frac{di}{dt} = C\frac{d^2U}{dt^2} + \frac{1}{R}\frac{dU}{dt} + \frac{1}{L}U.$$

Suppose $C = 0.3$ farads, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by

$$U(t) = e^{-0.06\pi t}\sin(2t - \pi).$$

If $i(0) = 0$, find the current i for the values $t = 0.1j$, $j = 0, 1, \ldots, 100$ using Euler's methods.

Problem 6. The following 2-step explicit formula

$$y_{n+1} = -4y_n + 5y_{n-1} + h(4f(t_n, y_n) + 2f(t_{n-1}, y_{n-1}))$$

applied to the problem $y' = f(t, y)$, $y(t_0) = y_0$ has the theoretical order of accuracy $p = 3$. Suppose we solve the initial value problem

$$y' = 0, \quad t \in [0, 1], \quad y(0) = y_0$$

(an exact solution $y(t) \equiv y_0$) using this method with the step size $h = 0.01$. Making plots of approximate solutions in the case of the following series of initial values
a) $y_0 = y_1 = 1$, $y_0 = y_1 = 1.5$, $y_0 = y_1 = 1.75$, $y_0 = y_1 = 1.875$, ...
b) $y_0 = y_1 = 1.3$, $y_0 = y_1 = 1.6$, $y_0 = y_1 = 1.7$, ...
we observe different behavior of approximate solutions. Try to explain the observed differences. Let you notice that in this particular case your algorithm is independent of the step value h.

Problem 7. Suppose we solve the initial value problem

$$y' = -100(y - \cos(t)) - \sin(t), \quad y(0) = 1, \quad t \in [0, 1],$$

by explicit and implicit Euler's methods. The exact solution of this problem is $y(t) = \cos(t)$. In both those cases determine the least value of the step size h for which the approximate solution at $t = 1$ differs from $y(1)$ less than by $\epsilon = 0.001$.

Problem 8. The initial value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad x(0) = 1, \quad y(0) = 0$$

has a solution $x(t) = 2e^{-t} - e^{-1000t}$, $y(t) = -e^{-t} + e^{-1000t}$. Solve it numerically using explicit and implicit Euler's methods. In each case determine the least value of the step size h when the approximation error at $t = 1$ is less than $\epsilon = 0.0001$.

Problem 9. Consider the following system of differential equations

$$\begin{cases} \dot{x} = ax - by \\ \dot{y} = bx + ay \end{cases}$$

a) Let $a = 0$. For each $b = 1$, 0.5, 0.1, and 0.05 draw the slope field of the system (you can use a Matlab command *quiver*).

b) Let $a = 0$, $b = 1$. Draw approximate solutions for initial value $[x(0), y(0)] = [0, 1]$ obtained by the explicit Euler method on the interval $t \in [0, 4\pi]$. Do this exercise for a number of step sizes of h, for example $h = 0.1$, 0.05, 0.025, 0.0125.

c) The exact solution in b) is $x(t) = -e^{at}\sin(bt), y(t) = e^{at}\cos(bt)$. Discuss the order of accuracy of the mehod.

d) Let $a = 0.2$ and $a = -0.2$. Repeat exercises b) and c).

Problem 10. Limit circle. Draw a number of approximate solutions to the following system of equations

$$\begin{cases} \dot{x} = -y + x\left[1 - (x^2 + y^2)^{\frac{1}{2}}\right] \\ \dot{y} = x + y\left[1 - (x^2 + y^2)^{\frac{1}{2}}\right]. \end{cases}$$

1. Draw the slope field for the system of equations.
2. You should observe that the system has an exact solution $(x(t), y(t)$ satisfying relation $x^2(t) + y^2(t) = 1$.

Problem 11. It is know that equation $\ddot{x} + \dot{x} - \text{sign}(\dot{x}) + x = 0$ has a periodic solution $(x(t), y(t))$, i.e. there exists a constant $T > 0$ such that $x(t+T) = x(t)$, $y(t+T) = y(t)$ for every $t \in R$. Try to justify this hypothesis.
a) Draw a number of approximate solutions obtained by the Runge-Kutta method illustrating existence of that periodic soluion.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - y + \text{sign}(y) \end{cases}$$