Numerical methods in differential equations 3

Solutions to the problems below should be prepared in the form of Matlab or Python codes with attached flowcharts explaining your algorithms.

Problem 1. Use the shooting algorithm to approximate the solution of the following boundary value problem with accuracy $\epsilon = 0.001$ at the right end of the interval. The actual solution is given for comparison to your result.

a) $y'' = \frac{1}{2}y^3, \;\; 1 \le x \le 2, \;\; y(1) = -\frac{2}{3}, \;\; y(2) = -1, \;\; y(x) = 2/(x-4).$

b) $y'' = y^3 - yy', \;\; 1 \le x \le 2, \;\; y(1) = \frac{1}{2}, \;\; y(2) = \frac{1}{3}, \;\; y(x) = 1/(x+1).$

c) $y'' = 2y^3 - 6y - 2x^3, \;\; 1 \le x \le 2, \;\; y(1) = 2, \;\; y(2) = \frac{5}{2}, \;\; y(x) = x + \frac{1}{x}.$

d) $y'' = -(y')^2 - y + \ln x, \;\; 1 \le x \le 2, \;\; y(1) = 0, \;\; y(2) = \ln 2, \;\; y(x) = \ln x.$

For approximation of the corresponding initial value problems use the Runge-Kutta method of order 2.

Problem 2. The maximum principle. Let L_h be an operator defined for finite sequences $\{U_j\}$, $j = 0, 1, \ldots, N$ by the formula

$$L_h U_j = -\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + q(x_j)U_j, \;\; j = 1, 2, \ldots, N-1$$

Show that if the sequence $\{U_j\}$, $j = 0, 1, \ldots, N$ is such that $L_h U_j \le 0$, ($L_h U_j \ge 0$) $j = 1, 2, \ldots, N-1$ and $q(x) \ge 0$, then

$$\max_j U_j = \max\{U_0, U_N, 0\} \;\; (\min_j U_j = \min\{U_0, U_N, 0\}).$$

Verify that observation in the case of the finite difference solution of the following problem

$-u'' + xu = 0$
$u(0) = 0, \;\; u(1) = 1.$

Problem 3. Use the finite difference method to approximate the following linear boundary value problem at the point of the uniform division $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The actual solution is given for comparison to your result.

a) $y'' = -4y' + 4y, \;\; 0 \le x \le 1,$
$y(0) = 1, \;\; y(1) = e^{-2+2\sqrt{2}}, \;\; y(x) = e^{(-2+2\sqrt{2})x},$

b) $y'' = -3y' + 2y + 2x + 3, \;\; 0 \le x \le 1,$
$y(0) = 2, \;\; y(1) = -4 + 5e^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}, \;\; y(x) = -3 + 5e^{(-\frac{3}{2}+\frac{\sqrt{17}}{2})x} - x,$

c) $-y'' - (x+1)y' + 2y = -(x^2+3)e^x, \;\; 0 \le x \le 1,$
$y(0) = -1, \;\; y(1) = 0, \;\; y(x) = (x-1)e^x,$

d) $-y''(x) + (x^2+1)y(x) = (\pi^2 + x^2 + 1)\sin(\pi x), \;\; 0 \le x \le 1,$
$y(0) = 0, \;\; y(1) = 0, \;\; y(x) = \sin(\pi x).$

1. Plots the graphs of the exact solution y_d and approximate solutions y_p at the knots $\{x_i\}_{0\le i\le n}$ for $n = 50, 100, 200, 400, 800$.
2. Observing the error

   $$\text{error}(h) = \max_{0\le i\le n} |y_{p,i} - y_{d,i}|$$

   $n = 50, 100, 200, 400, 800$ try to determine the accuracy order of the method.

Problem 4. Use the Ritz-Galerkin method to approximate the solution to each of the following boundary value problems. The actual solution $y_d(x)$ is given for comparison purposes.

$$1)\quad \begin{cases} -y''(x) + e^x y(x) = (\pi^2 + e^x)\sin(\pi x) \\ y(0) = 0, \quad y(1) = 0, \quad y_d(x) = \sin(\pi x), \end{cases}$$

$$2)\quad \begin{cases} -((x+2)y')' + (x^2+1)y = -\sin(\pi x) - x\cos(\pi x)\pi - (x+2) \\ \quad (2\cos(\pi x)\pi - x\sin(\pi x)\pi^2) + (x^2+1)x\sin(\pi x) \\ y(0) = 0, \quad y(1) = 0, \quad y_d = x\sin(\pi x) \end{cases}$$

$$3)\quad \begin{cases} -y''(x) + (x+1)y(x) = (\pi^2 + x + 1)\sin(\pi x) \\ y(0) = 0, \quad y(1) = 0, \quad y_d(x) = \sin(\pi x), \end{cases}$$

$$4)\quad \begin{cases} -((x+1)y')' + x^2 y = -\cos(\pi x)\pi + (x+1)\sin(\pi x)\pi^2 + x^2\sin(\pi x) \\ y(0) = 0, \quad y(1) = 0, \quad y_d = \sin(\pi x), \end{cases}$$

As test functions take the piecewise linear continuous functions. The space $S_0^1(\Delta)$ of such functions can be described as follows: Δ is a uniform partition of the interval $[a,b]$, $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, and the basis of $S_0^1(\Delta)$ consists of the hat functions $\phi_i(x) = \phi(\frac{x-x_i}{h})$, $i = 1, 2, \ldots, n$, where the mother function is defined as

$$\phi(x) = \begin{cases} -|x| + 1, & -1 \le x \le 1 \\ 0, & x \notin [-1,1]. \end{cases}$$

1. Plots the graphs of the exact solution $y_d(x)$ and approximate solutions $y_p(x)$ for $n = 50, 100, 200, 400, 800$.
2. Observing the error

   $$\text{error}_2(h) = \left(\int_a^b (y_d(x) - y_p(x))^2 dx\right)^{1/2}$$

   $n = 50, 100, 200, 400, 800$ try to determine the accuracy order of the method.

Problem 5. Some approximation properties of the space $S_0^1(\Delta)$ (see the exercise above). Let $v(x) \in C[a,b]$, $v(a) = v(b)$. We define the interpolant $w_h(x) = I_h v(x) \in S_0^1(\Delta)$ of $v(x)$ by the relation

$$w_h(x_j) = v(x_j), \quad j = 0, 1, \ldots, n$$

in other words

$$w_h(x) = \sum_{j=1}^{n} v(x_j)\phi_j(x).$$

Show that there exists a constant C such that for all $K_j = [x_{j-1}, x_j]$

$$a) \;\; ||I_h v - v||_{K_j} = \left(\int\limits_{x_{j-1}}^{x_j} (I_h v(x) - v(x))^2 dx \right)^{1/2} \leq Ch^2 \left(\int\limits_{x_{j-1}}^{x_j} v''(x)^2 dx \right)^{1/2} .$$

$$b) \;\; ||(I_h v)' - v'||_{K_j} = \left(\int\limits_{x_{j-1}}^{x_j} ((I_h v)'(x) - v'(x))^2 dx \right)^{1/2} \leq Ch \left(\int\limits_{x_{j-1}}^{x_j} v''(x)^2 dx \right)^{1/2} .$$

$$(Hint : (I_h v)'(x) - v'(x) = \frac{1}{h} \int\limits_{x_{j-1}}^{x_j} v'(y) - v'(x) dy.)$$

Conclusion

$$||I_h v - v|| \leq Ch^2 \left(\int\limits_a^b v''(x)^2 dx \right)^{1/2} , \quad ||(I_h v)' - v'|| \leq Ch \left(\int\limits_a^b v''(x)^2 dx \right)^{1/2} .$$

Illustrate the result for $v(x) = \sin x$, $x \in [0, \pi]$ taking consecutively $h = 2^{-4}\pi, 2^{-5}\pi, 2^{-6}\pi, 2^{-7}\pi, \ldots$.