Lecture 2b

Slope field for the system of differential equations

(1) $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

Vector field $\vec{V}(x,y) = [f(x,y), g(x,y)]$, $(x,y) \in R^2$ is called the slope field for (1). To draw it we first normalize it which results in

$$\vec{V}(x,y) = \frac{1}{\sqrt{f(x,y)^2 + g(x,y)^2}}[f(x,y), g(x,y)].$$

To obtain a final picture we can use a Matlab command 'quiver'.

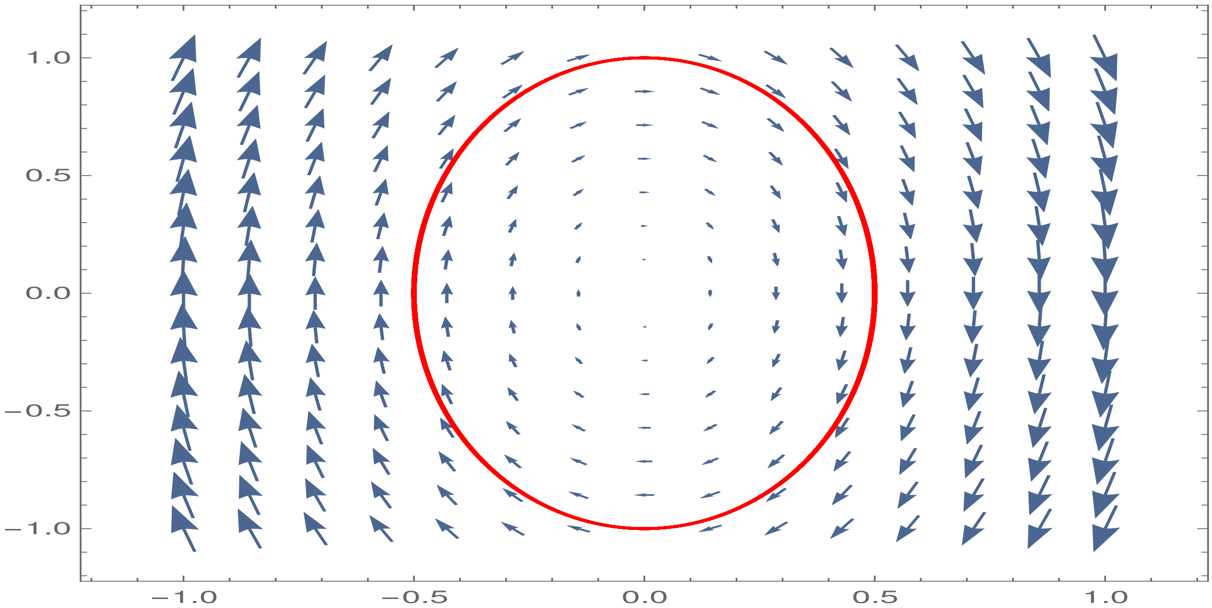
Example 1. Let us consider equation $z'' + 4z = 0$. We introduce the auxilliary functions $x = z$, $y = z'$ and obtain the system of equations

(2) $\begin{cases} x' = y \\ y' = -4x \end{cases}$

or in the matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the initial values $x(0) = 1/2$, $y(0) = 0$ we get the exact solution $x(t) = 1/2\cos(2t)$, $y(t) = -\sin(2t)$. The solution and the slope field are demonstrated in the picture below.



II. Stiffness in differential problems.

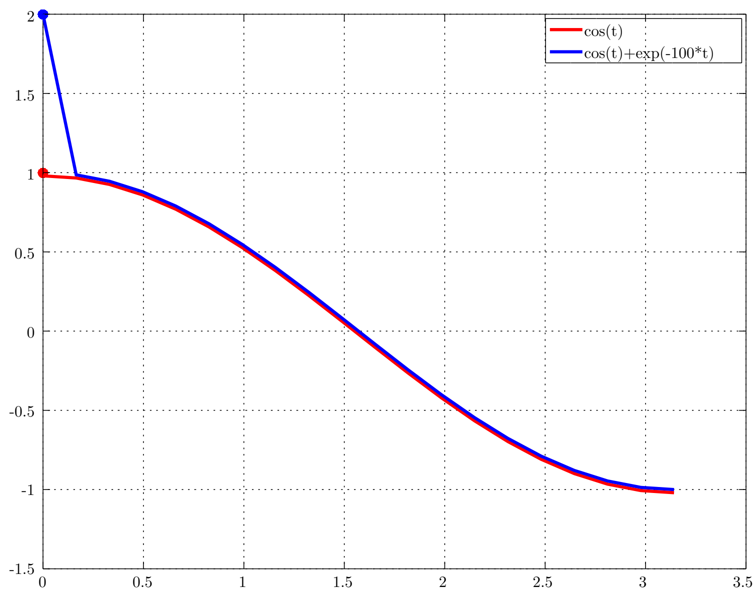
Example 2.

$$\begin{cases} y' = -100(y - \cos t) - \sin t \\ y(0) = 1 \end{cases}, \quad \text{the exact solution is a function } y(t) = \cos t$$

The general solution of the differential equation is

$$y(t) = \cos t + Ce^{-100t}$$

Any such a solution is rapidly shooting towards the curve $\cos t$. Applying the explicit method for obtaining the approximate solution we see that it reflects rather the general solution than the particular one, $\cos t$. However this requires the mesh size h satisfying the inequality $|1 - 100h| < 1$ (compare with equation $y' = -100y$), which gives $h < \frac{2}{100}$.



We observe a similar phenomenon for the system of equations

$$\begin{cases} x' = 998x + 1998y \\ y' = -999x - 1999y \end{cases}$$

with a solution $x(t) = -e^{-1000t} + 2e^{-t}$, $y(t) = e^{-1000t} - e^{-t}$. In the solution we note the disturbing term which can be neglected for not too small t's, but the Euler's explicit method reflects rather the full solution. However this requires the mesh size h satisfying $|1 - 1000h| < 1$, which gives $h < \frac{2}{1000}$.

III. Approximate solution of the boundary value problem for the second order differential equation

First, let us compare the initial value problem with the corresponding boundary value one.

$$(1)\quad \begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha, \quad y'(a) = \beta \end{cases}$$

$$(2)\quad \begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

The problem (1) has always a unique solution. In contrast to (1), the problem (2) sometimes has no solution or sometimes has an infinite number of solutions.

Example 3.

$$\begin{cases} y'' = -y \\ y(0) = 0, \quad y(\pi) = 1 \end{cases}$$

the general solution of this equation is $y(t) = A\cos t + B\sin t$.

Then we have $y(0) = A\cos 0 + B\sin 0 = A$, $y(\pi) = -A$. Thus the boundary condition causes that there are no A, B such that $y(t)$ would satisfy this condition. Therefore this boundary problem has no solutions.

Example 4.

$$\begin{cases} y'' = -y \\ y(0) = 0, \quad y(\pi) = 0 \end{cases}$$

the general solution of the equation is $y(t) = A\cos t + B\sin t$

Now $y(t) = B\sin t$, for any $B \in R$ is a solution of the problem. So the problem has a infinitly many solutions.

Examples of the boundary value problems comes frequently from the technical problems.

Example 5. A common problem in statics concerns the deflection of the elastic beam subjected to uniform loading, while the ends of the beam are fixed

$$\begin{cases} y'' = \frac{S}{EI}y + \frac{qx}{2EI}(x-l) \\ y(a) = 0, \quad y(b) = 0 \end{cases}$$

l is the length of the beam, q is the intensity of the uniform loading, $E(x)$ is a modulus of the elasticity at the point x, S is a stress at the ends of the beam, $I(x)$ is the central moment of inertia.

Example 6. The general form of the equations, which are known that the boundary value problem has a unique solution is as follows

$$\begin{cases} -(p(x)y')' + q(x)y = f(x), \\ y(a) = \alpha, \quad y(b) = \beta. \end{cases}$$

$$p(x) \geq p_0 > 0, \quad q(x) \geq 0$$
$$p(x) \in C^1[a,b], \quad q(x) \in C[a,b]$$

The shooting method for the approximate solution of the boundary problem

$$\begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases} \tag{3}$$

We assume the problem (3) has a unique solution.

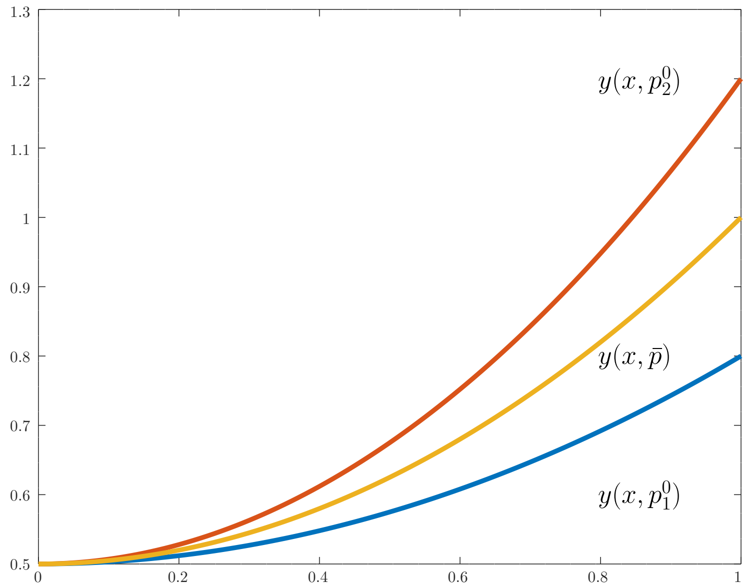
We will consider an auxilliary initial value problem

$$\begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha, \quad y'(a) = p \end{cases} \tag{4}$$

where p is a parameter. The solution of (4) is denoted by $y(t;p)$.

The construction of the approximate solution. First we find experimentally two parameters p_0^0, p_1^0 such that

$$y(b;p_0^0) < \beta, \quad y(b;p_1^0) \geq \beta$$



Let us assume $p_0^0 < p_1^0$. We are looking for a $\bar{p}$ such that $y(b, \bar{p}) = \beta$ and $p_0^0 \leq \bar{p} \leq p_1^0$.

2. Let

$$c = \frac{1}{2}(p_0^0 + p_1^0).$$

We define new parameters p_0^1 and p_1^1 in the following way

$$\begin{cases} p_0^1 = c, \quad p_1^1 = p_1^0, & \text{if } y(b;c) < \beta \\ p_0^1 = p_0^0, \quad p_1^1 = c, & \text{if } y(b;c) > \beta \end{cases}$$

3. at the n-th level we have p_0^n, p_1^n, passing to the $n+1$-th level we take

$$\begin{cases} p_0^{n+1} = c, \quad p_1^{n+1} = p_1^n, & \text{if } y(b;c) < \beta \\ p_0^{n+1} = p_0^n, \quad p_1^{n+1} = c, & \text{if } y(b;c) > \beta \end{cases}$$

Thus we have constructed the sequences $\{p_0^n\}$ and $\{p_1^n\}$, for which we have

1. $|p_1^{n+1} - p_0^{n+1}| = \dfrac{1}{2}|p_1^n - p_0^n|$
2. $p_0^0 \leq p_0^n \leq p_0^{n+1} \leq \cdots \leq p_1^{n+1} \leq p_1^n \leq p_1^0$

From 1. it follows that

$$|p_1^n - p_0^n| = \frac{1}{2^n}|p_1^0 - p_0^0| = \frac{1}{2^n}d.$$

Hence there exists p^* such that

$$\lim p_0^n = p^* = \lim p_1^n.$$

We note that $y(t;p^*)$ is the exact solution. As an approximate solution at the n-th step we take $y(t;c)$, where $c = (p_0^n + p_1^n)/2$.

Remark. In the numerical solution instead of the exact solution $y(x;c)$ we have to operate with its approximation usually obtained by a Runge-Kutta type method.