Lecture 3

Topics

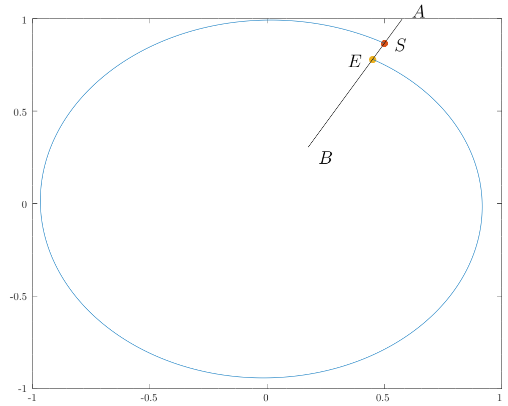
1. Determining periodic solutions
2. Finite difference method for the approximate solution of boundary value problems for second order differential equations

Determining the periodic solutions.
Let us consider the system of differential equations

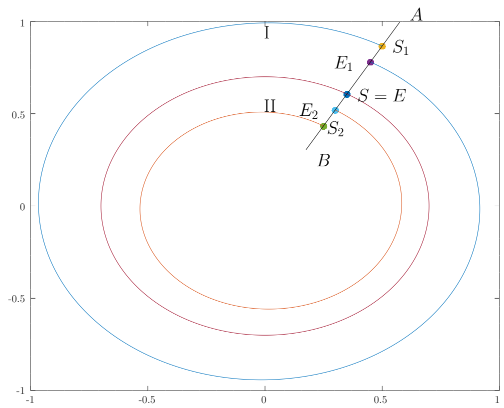
$$(1) \qquad \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

where a pair of functions $(x(t), y(t))$, $t \in \mathbb{R}$ is a solution. We assume that the functions f i g are regular. Under these assumptions any initial value problem $(x(0) = x_0, y(0) = y_0)$ for the system (1) has a unique solution. Hence it follows that the graphs of two solutions either concide or are disjoint.

Let AB be a segment in the plane. For any point $S \in AB$ (the start point) we construct a solution to the system such that $(x(0), y(0)) = S$.
We continue this construction until the solution reaches the segment once more. The point, where the solution crosses the segment is denoted by E (the end point).



If we are able to construct two solutions with the start points S_1, S_2 and the end points E_1, E_2 respectively in the order as in the picture below, then there exists a solution with a start point $S \in AB$, whose end point $E = S$. This solution is a closed curve and consequently it is a sought periodic solutions, i.e. there exists $T > 0$ such that $x(t+T) = x(t)$, $y(t+T) = y(t)$ for $t \in \mathbb{R}$.

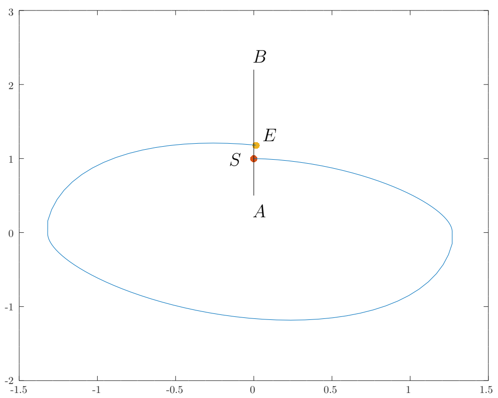


Example. Let you detect experimentaly a periodic solution to the second order equation $x'' + x - \text{sign}(x') + x' = 0$.

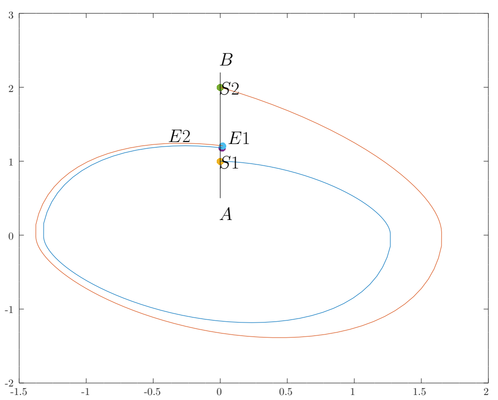
Solution. First we transform the equation to the system of equations

$$(2) \quad \begin{cases} x' = y \\ y' = -x + \text{sign}(y) - y. \end{cases}$$

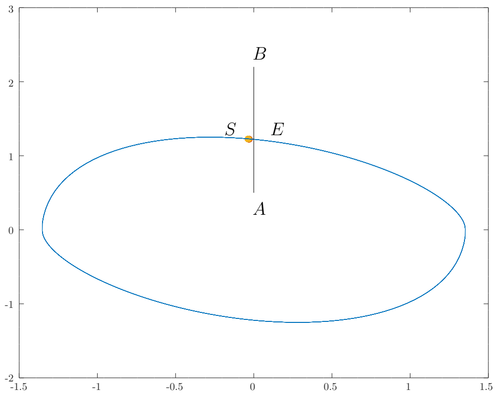
We choose the segment AB, where $A = (0, 0.5)$, $B = (0, 2.2)$ and construct a number of approximate solutions to initial value problems with initial points $S \in AB$. We continue the construction until the solution reaches the segment AB. The approximate solution are constructed by the second order Runge-Kutta's method.



Our aim is to find two solutions for which the following order of the points in the segment: A, S_1, E_1, E_2, S_2, B is reached. We choose two solutions corresponding to the initial points $S_1 = (0, 1)$ and $S_2 = (0, 2)$.



It follows from the theory of differential equation (the Poincare-Bendixon theorem) that there exists a periodic solution with the initial point S in the segment E_1E_2.


B
S
E
A

The finite difference method for the approximate solution of the boundary value problem

$$(3)\quad \begin{cases} -y'' + q(x)y = g(x), & x \in (a,b) \\ y(a) = \alpha \quad y(b) = \beta \end{cases}$$

where $q(x) \geq 0$ and $g(x)$ are continuous.
It is known that such a problem has a unique solution.

The construction of the approximate solution.

Let us consider the partition: $a = x_0 < x_1 < \dots x_N = b$, $x_j = a + jh$, $h = \frac{b-a}{N}$.

Algorithm

We construct a sequence $\{y_i\}$, whose elements are considered as approximate values $y_i \approx y(x_i)$, $i = 0, 1, 2 \dots, N$ by the following algorithm

$$\begin{aligned} & y_0 = \alpha \\ & -\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + q_i y_i = g_i, \quad i = 1, 2, \dots, N-1 \\ & y_N = \beta \end{aligned}$$

or

$$\begin{aligned} & (2 + q_1 h^2)y_1 - y_2 = g_1 h^2 + \alpha, \\ & -y_{i-1} + 2y_i - y_{i+1} + q_i h^2 y_i = g_i h^2, \quad i = 2, 3, \dots, N-2 \\ & -y_{N-2} + (2 + q_{N-1}h^2)y_{N-1} = g_{N-1}h^2 + \beta, \end{aligned}$$

To simplify calculations, we introduce

$$(4)\quad A = \begin{bmatrix} 2+q_1h^2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2+q_2h^2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2+q_3h^2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2+q_{N-2}h^2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2+q_{N-1}h^2 \end{bmatrix}$$

$$\bar{y} = [y_1, y_2, \ldots, y_{N-1}]^T,$$
$$\bar{b} = [g_1 h^2 + \alpha, g_2 h^2, \ldots, g_{N-2} h^2, g_{N-1} h^2 + \beta]^T$$

As a result we obtain a system of the linear equations to solve

$$A\bar{y} = \bar{b}. \quad (5)$$

The existence of its solutions is guaranted by the following properties of A.

The matrix A is positively definite (for sufficiently small $h > 0$) i.e. one proves that the inequality $x^T \cdot Ax > 0$ is satisfied for any vector $x \neq 0$, $x \in R^{N-1}$.

Hence it follows that there exists no vector $x \neq 0$ such that $Ax = 0$, in other words the matrix A is nonsingular ($\det A \neq 0$).

Corollary. The approximate problem has exactly one solution for sufficiently small $h > 0$.

The error estimate is given in the following theorem

Twierdzenie 1. *If* $q(x) \geq 0$*, then the exact boundary value problem* (1) *has exactly one solution* $y(x)$*. Similarly the approximate problem* (3) *has exactly one solution (for sufficiently small* $h > 0$*). Moreover, if* $y(x) \in C^4[a, b]$*, then the following estimate holds*

$$|y(x_i) - y_i| \leq \frac{Mh^2}{24}(x_i - a)(b - x_i)$$

for $i = 1, 2, \ldots, N - 1$*, where* $M = \sup |y^{(4)}(x)|$.

Motivation for the form of the finite difference algorithm

As an approximation to $y''(x_i)$, we take

$$\Delta^2 y(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$$

Let us assume $y(x) \in C^4[a,b]$ and let us estimate the error $\tau_i(y) = y''(x_i) - \Delta^2 y(x_i)$. Using the Taylor's formula we obtain

$$y(x_i \pm h) = y(x_i) \pm hy'(x_i) + \frac{h^2}{2}y''(x_i) \pm \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y^{(4)}(x_i \pm \Theta_i^{\pm} h), \quad 0 < \Theta_i^{\pm} < 1.$$

Hence

$$\Delta^2 y(x_i) = y''(x_i) + \frac{h^2}{24}[y^{(4)}(x_i + \Theta_i^{+} h) + y^{(4)}(x_i - \Theta_i^{-} h)] = y''(x_i) + \frac{h^2}{12}y^{(4)}(x_i + \Theta_i h)$$

and finally

$$\tau_i(y) = y''(x_i) - \Delta^2 y(x_i) = -\frac{h^2}{12}y^{(4)}(x_i + \Theta_i h), \quad |\Theta_i| < 1.$$

To summarize, since $y''(x_i) = \Delta^2 y(x_i) + \tau_i(y)$, we observe that the following relations hold

$$y(x_0) = \alpha;$$
$$-\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} + q(x_i)y(x_i) = g(x_i) + \tau_i(y), \quad i = 1, 2, \ldots, N-1$$
$$y(x_N) = \beta$$

The finite difference method for the approximate solution of more general boundary value problems

$$(6)\quad \begin{cases} -y'' + p(x)y' + q(x)y = g(x), & x \in (a,b) \\ y(a) = \alpha \quad y(b) = \beta \end{cases}$$

Assuming this equation has a unique solution we descritize the problem similarly as problem (3) and obtain

$$\begin{aligned} &y_0 = \alpha \\ &-\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = g_i, \quad i = 1, 2, \ldots, N-1 \\ &y_N = \beta \end{aligned}$$

or

$$\begin{aligned} &(2 + q_1 h^2) y_1 - (1 - p_1 h/2) y_2 = g_1 h^2 + (1 + p_1 h/2)\alpha, \\ &-(1 + p_i h/2) y_{i-1} + (2 + q_i h^2) y_i - (1 - p_i h/2) y_{i+1} = g_i h^2, \quad i = 2, 3, \ldots, N-2 \\ &-(1 + p_{N-1} h/2) y_{N-2} + (2 + q_{N-1} h^2) y_{N-1} = g_{N-1} h^2 + (1 - p_{N-1} h/2)\beta, \end{aligned}$$

where $p_i = p(x_i)$, $q_i = q(x_i)$ and $g_i = (g(x_i)$, $i = 1, 2, \ldots, N$. To simplify the algorithm, we introduce

$$A = \begin{bmatrix} 2 + q_1 h^2 & -(1 - p_1 h/2) & 0 & \ldots & 0 & 0 \\ -(1 + p_2 h/2) & 2 + q_2 h^2 & -(1 - p_2 h/2) & \ldots & 0 & 0 \\ 0 & -(1 + p_3 h/2) & 2 + q_3 h^2 & -(1 - p_3 h/2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -(1 + p_{N-2} h/2) & 2 + q_{N-2} h^2 & -(1 - p_{N-2} h/2) \\ 0 & 0 & 0 & 0 & -(1 + p_{N-1} h/2) & 2 + q_{N-1} h^2 \end{bmatrix}$$

$$\begin{aligned} &\bar{y} = [y_1, y_2, \ldots, y_{N-1}]^T, \\ &\bar{b} = [g_1 h^2 + (1 + p_1 h/2)\alpha, g_2 h^2, \ldots, g_{N-2} h^2, g_{N-1} h^2 + (1 - p_{N-1} h/2)\beta]^T \end{aligned}$$

As a result we obtain a system of the linear equations to solve

$$(7)\quad A\bar{y} = \bar{b}.$$