Lecture 4

Topics

1. The Ritz-Galerkin and the finite element method for the boundary value problem for the second order differential equations

The Ritz-Galerkin-Rayleigh method for the problem

$$(1) \qquad \begin{cases} -(p(x)y')' + q(x)y = f(x), & x \in (a,b) \\ y(a) = 0, \quad y(b) = 0. \end{cases}$$

Assumptions: $p(x) \in C^1[a,b]$, $p(x) \geq p_0 > 0$, $q(x)$, $f(x) \in C[a,b]$, $q(x) \geq 0$. Under these assumptions the problem has a unique solution.

1. Standard (or regular, classical) solutions. The function $y(x) \in C^2[a,b]$ which satisfies all the conditions in the problem above is a standard (regular) solution.

2. Weak solutions.

The introductory observation. Let

$$v \in C^1[a,b],\ v(a) = 0,\ v(b) = 0 \ \text{ (we will write } v \in \mathcal{H}_0^1(a,b))$$

Then a function $y(x) \in C^2[a,b]$ is a solution of (1), if and only if

$$(2) \qquad \int_a^b [-(p(x)y')' + q(x)y]v\,dx = \int_a^b fv\,dx$$

for every $v \in \mathcal{H}_0^1(a,b)$. Since

$$\int_a^b -(p(x)y')'v\,dx = -p(x)y'v|_a^b + \int_a^b p(x)y'v'\,dx = \int_a^b p(x)y'v'\,dx,$$

condition (2) can be replaced by

$$(3) \qquad \int_a^b p(x)y'v' + q(x)yv\,dx = \int_a^b fv\,dx$$

This relation is a basis for the definition of a weak solution.

Definition. The function $y \in \mathcal{H}_0^1(a,b)$ is a weak solution of (1) if and only if the condition (3) holds for every $v \in \mathcal{H}_0^1(a,b)$. Note that y is required to be only a C^1 regular function, so it may not satisfy the boundary problem (1) in a classical sense.

For any $v, w \in \mathcal{H}_0^1(a,b)$ we define

$$B(u,v) = \int_a^b p(x)u'v' + q(x)uv\,dx, \quad (f,v) = \int_a^b fu\,dx,$$

We observe that

1. $B(u, v) = B(v, u)$
2. $B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v)$
3. $B(u, u) > 0$ for $u \neq 0$.
4. $(f, \alpha v_1 + \beta v_2) = \alpha(f, v_1) + \beta(f, v_2)$

Hence $B(u, v)$ is a bilinear form and (f, v) is a linear one. Using these forms we can define a weak solution $y \in \mathcal{H}_0^1(a, b)$ to the original problem as a function satisfying

$$B(y, v) = (f, v) \quad \text{for every } v \in \mathcal{H}_0^1(a, b).$$

3. Variational solutions.

We define the energy functional

$$E(u) = \frac{1}{2}B(u,u) - (f,u),$$

for every $u \in \mathcal{H}_0^1(a,b)$.

and the variational solution of the problem

Definition. The function $y \in \mathcal{H}_0^1(a,b)$ is a variational solution of (1) if and only if

$$E(y) = \min_{u \in \mathcal{H}_0^1(a,b)} E(u).$$

A comparison of the weak and variational solutions is given in the following theorem

Twierdzenie 1. *The function $y \in \mathcal{H}_0^1(a,b)$ is a variational solution if and only if it is a weak solution to the problem* (1).

Proof. First we assume that $y(x)$ is a variational solution. Then for any $v \in \mathcal{H}_0^1(a,b)$ and $t \in R$ we have $y + tv \in \mathcal{H}_0^1$ and

$$E(y) \le E(y+tv) = \frac{1}{2}B(y+tv, y+tv) - (f, y+tv) =$$
$$\frac{1}{2}B(y,y) - (f,y) + t(B(y,v) - (f,v)) + t^2B(v,v) =$$
$$E(y) + t(B(y,v) - (f,v)) + t^2B(v,v).$$

Hence, we have

$$0 \le t(B(y,v) - (f,v)) + t^2B(v,v) \le t(B(y,v) - (f,v) + tB(v,v))$$

Let us consider $t \to 0+$. Then we have

$$B(y,v) - (f,v) \ge -tB(v,v) \text{ with } -tB(v,v) \to 0.$$

Let us consider $t \to 0-$. Then we have

$$B(y,v) - (f,v) \le -tB(v,v), \text{ with } -tB(v,v) \to 0.$$

Finally, we get

$$B(y,v) = (f,v) \text{ or } \int_a^b p(x)y'v' + q(x)yvdx = \int_a^b fvdx,$$

for any $v \in \mathcal{H}_0^1(a,b)$. This means that $y(x)$ is a weak solution.

Now we assume that $y(x)$ is a weak solution, i.e.

$$B(y,v) - (f,v) = 0.$$

Then for any $w \in \mathcal{H}_0^1$ we also have $v = w - y \in \mathcal{H}_0^1$ and

$$E(w) = E(y+v) = \underbrace{\frac{1}{2}B(y,y) - (f,y)}_{} + \underbrace{(B(y,v) - (f,v))}_{} + \underbrace{\frac{1}{2}B(v,v)}_{} =$$

$$E(y) + 0 + \frac{1}{2}B(v,v) \geq E(y),$$

which means that the energy functional reaches its minimum at $y(x)$. This ends the proof.

Construction of an approximate solution.

Let $V_m \subset \mathcal{H}_0^1$ be a linear finite dimensional subspace with basis functions $\phi_1(x)$, $\phi_2(x)$, ... $\phi_m(x)$. We define an approximate solution as a function

$$(4) \quad y_m(x) = \sum_{i=1}^{m} \alpha_i \phi_i(x) \in V_m \subset \mathcal{H}_0^1$$

such that

$$(5) \quad B(y_m, v_m) = (f, v_m),$$

is satisfied for any $v_m(x) \in V_m$ or in other words:

y_m is a weak solution in V_m.

Let you compare definition of the approximate solution with defifinition of the exact one:

$$B(y, v) = (f, v) \quad \text{for every } v \in \mathcal{H}_0^1(a, b).$$

Solution of an approximate problem.

Our aim is to determine coefficients α_1, α_2,..., α_m. Taking in (5) as $v_m(x)$, subsequently the basis functions $\phi_i(x)$, $i = 1, 2, \ldots, m$ we obtain the system of the linear equations

$$(6) \quad \sum_{j=1}^{m} a_{ij}\alpha_j = f_i, \quad i = 1, 2, \ldots, m$$

where

$$a_{ij} = B(\phi_i, \phi_j) = \int_a^b p(x)\phi_i'(x)\phi_j'(x) + q(x)\phi_i(x)\phi_j(x)dx,$$

$$f_j = (f, \phi_j) = \int_a^b f\phi_j dx.$$

We will write the equations above in the following matrix form

$$(7) \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

Matrix $[B(\phi_i, \phi_j)]$ is called a *stiffness matrix* and vector $(f_1, f_2, \dots, f_m)$ is called a *load vector.*

Equivalently, system of equations (6) (or (7)) can be obtained also in the following way. We observe that y_m in (5) satisfies the variational condition

$$(8) \quad E(y_m) = \min_{v_m \in V_m} E(v_m).$$

Formula (8) is expressed in terms of functions y_m and v_m. Our aim is to express it in terms of numbers. Since

$$v_m(x) = \sum_{i=1}^{m} \beta_i \phi_i(x)$$

where β_1, β_2,..., β_m are some coefficients, we have

$$E(v_m) = E(\beta_1, \beta_2 \dots, \beta_m) = \frac{1}{2} B(y_m, y_m) - (f, y_m) =$$
$$\frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \beta_i \beta_j - \sum_{i=1}^{n} f_i \beta_i,$$

Now, variational formula (8) can be rewritten as follows

$$(9) \quad E(\alpha_1, \alpha_2, \dots, \alpha_m) = \min_{\beta_1, \beta_2, \dots, \beta_m} E(\beta_1, \beta_2, \dots, \beta_m)$$

The solution of the above ekstremum problem is obtained by analytic methods. Since

$$\frac{\partial E}{\partial \beta_k} = \frac{\partial}{\partial \beta_k} \left(\frac{1}{2} \sum_{i,j=1}^{m} a_{ij} \beta_i \beta_j - \sum_{i=1}^{n} f_i \beta_i \right) = \sum_{j=1}^{m} a_{kj} \beta_j - f_k, \quad k = 1, 2, \dots, m$$

hence it follows that $E(\beta_1, \beta_2, \dots, \beta_m)$ reaches its minimum at $(\alpha_1, \alpha_2, \dots, \alpha_m)$, if

$$\frac{\partial E}{\partial \beta_k}(\alpha_1, \alpha_2, \dots, \alpha_m) = 0, \quad k = 1, 2, \dots, m.$$

It is easily seen that these conditions form simply system of equations (7).

The analysis of the system (7).

Since

$$B(\phi_i, \phi_j) = B(\phi_j, \phi_i),$$

we have $a_{ij} = a_{ji}$ which means that the stiffness matrix is symmetric. Moreover it is positive definite which means that

$$\sum_{i,j=1}^{m} a_{ij}\beta_i\beta_j > 0$$

for any system of coefficients $(\beta_1, \beta_2, \ldots, \beta_m) \neq 0$. In such a case the system (7) has exactly one solution $(\alpha_1, \alpha_2, \ldots, \alpha_m)$, so there exists a uniquely defined approximate solution

$$y_m(x) = \sum_{i=1}^{m} \alpha_i \phi_i(x).$$

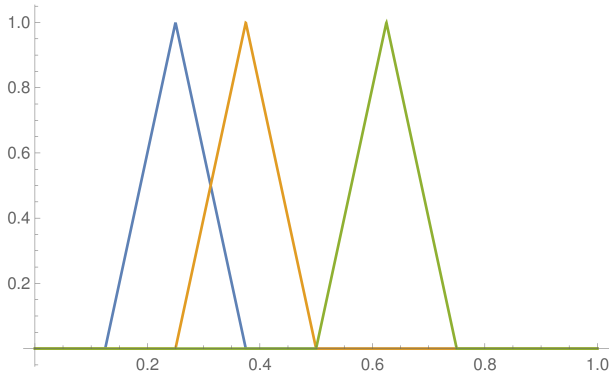
The construction of an approximate solution.

Example 1.

The first step is to form a uniform partition of $[a, b]$ by choosing points $a = x_0 < x_1 < \cdots < x_n = b$ with $h = x_{i+1} - x_i = \frac{b-a}{n}$, $i = 0, 1, \ldots, n-1$. We define the basis functions as

$$\phi_i(x) = \begin{cases} 0, & a \leq x \leq x_{i-1}, \\ \frac{1}{h}(x - x_{i-1}), & x_{i-1} \leq x \leq x_i, \\ \frac{1}{h}(x_{i+1} - x), & x_i \leq x \leq x_{i+1}, \\ 0, & x_{i+1} \leq x \leq b, \end{cases}$$

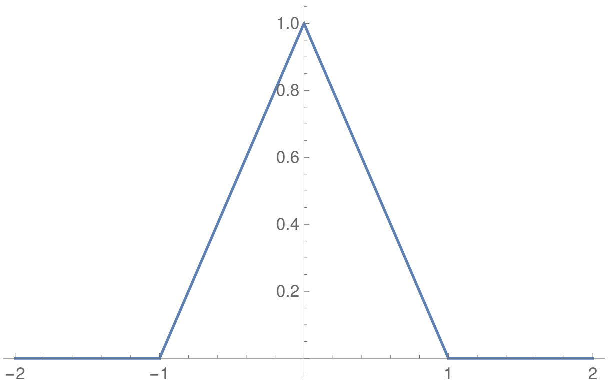
for $i = 1, 2, 3, \ldots, n-1$.



Remark. There is also another way of construction of the functions $\phi_i(x)$ above. Namely, let us take a "mother" function

$$\phi(x) = \begin{cases} 1 - |x|, & -1 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

and let us define $\phi_i(x) = \phi\left(\frac{x - x_i}{h}\right)$, $i = 1, 2, , 3, \ldots, n-1$.



pic. the "mother" function $\phi(x)$

Since functions $\phi_i(x)$ are piecewise linear, the derivatives $\phi_i'(x)$, while not continuous, are constant on the open subinterval (x_j, x_{j+1}) for each $j = 0, 1, \ldots, n-1$. Thus we have

$$\phi_i'(x) = \begin{cases} 0, & a \le x \le x_{i-1}, \\ \frac{1}{h}, & x_{i-1} \le x \le x_i, \\ -\frac{1}{h}, & x_i \le x \le x_{i+1}, \\ 0, & x_{i+1} \le x \le b, \end{cases}$$

$i = 1, 2, 3, \ldots, n-1.$

Since ϕ_i and ϕ_i' are nonzero only on (x_{i-1}, x_{i+1}), we observe that

$$\phi_i(x)\phi_j(x) \equiv 0 \text{ and } \phi_i'(x)\phi_j'(x) \equiv 0, \text{ if } |i-j| > 1.$$

As a consequence, linear system (7) reduces to an $n \times n$ tridiagonal linear system with the stiffness matrix entries $a_{ij} = 0$ for $|i-j| > 1$, while the nonzero entries are as follows

$$\begin{aligned} & a_{ii} = \int_a^b p(x)\phi_i'(x)^2 + q(x)\phi_i(x)^2 dx = \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} p(x)dx + \\ (10) \quad & \frac{1}{h^2}\int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 q(x)dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 q(x)dx \approx \\ & \frac{1}{h^2}(p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}}))h + \frac{1}{h^2}(q(x_{i-\frac{1}{2}}) + q(x_{i+\frac{1}{2}}))\frac{h^3}{3} = \\ & \left(p(x_{i-\frac{1}{2}}) + p(x_{i+\frac{1}{2}})\right)\frac{1}{h} + \left(q(x_{i-\frac{1}{2}}) + q(x_{i+\frac{1}{2}})\right)\frac{h}{3} \end{aligned}$$

for each $i = 1, 2, \ldots, n-1$, $x_{i-\frac{1}{2}} = x_i - \frac{h}{2}$, $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$.

$$
\begin{aligned}
&a_{i,i+1} = \int_a^b p(x)\phi_i'(x)\phi_{i+1}'(x) + q(x)\phi_i(x)\phi_{i+1}(x)dx = \\
&-\frac{1}{h^2}\int_{x_i}^{x_{i+1}} p(x)dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x - x_i)(x_{i+1} - x)q(x)dx \approx \\
&-\frac{1}{h^2}p(x_{i+\frac{1}{2}})h + \frac{1}{h^2}q(x_{i+\frac{1}{2}})\frac{h^3}{6} = -\frac{1}{h}p(x_{i+\frac{1}{2}}) + q(x_{i+\frac{1}{2}})\frac{h}{6}
\end{aligned}
$$

for each $i = 1, 2, \ldots, n-2$;

$$
\begin{aligned}
&a_{i+1,i} = \int_a^b p(x)\phi_{i+1}'(x)\phi_i'(x) + q(x)\phi_{i+1}(x)\phi_i(x)dx = \\
&-\frac{1}{h^2}\int_{x_i}^{x_{i+1}} p(x)dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x - x_i)(x_{i+1} - x)q(x)dx \approx \\
&-\frac{1}{h^2}p(x_{i+\frac{1}{2}})h + \frac{1}{h^2}q(x_{i+\frac{1}{2}})\frac{h^3}{6} = -\frac{1}{h}p(x_{i+\frac{1}{2}}) + q(x_{i+\frac{1}{2}})\frac{h}{6}
\end{aligned}
$$

for each $i = 1, 2, 3, \ldots, n-2$. The entries f_i are given by

$$
\begin{aligned}
&f_i = \int_a^b f(x)\phi_i(x)dx = \\
&\frac{1}{h}\int_{x_{i-1}}^{x_i} f(x)(x - x_{i-1})dx + \frac{1}{h}\int_{x_i}^{x_{i+1}} f(x)(x_{i+1} - x)dx \approx \\
&\frac{1}{h}\left(f(x_{i-\frac{1}{2}})\frac{h^2}{2} + f(x_{i+\frac{1}{2}})\frac{h^2}{2}\right) = 0.5h\left(f(x_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}})\right)
\end{aligned}
$$

A practical guide for approximate calculating the integrals above over the intervals (x_{i-1}, x_i) and (x_i, x_{i+1}): we have replaced functions $p(x)$, $q(x)$ and $f(x)$ with their values taken at the points $x_{i-\frac{1}{2}} = x_i - \frac{h}{2}$ and $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$ and further we calculate the integrals accurately. It is possible to use other approximate methods of integration based on the following quadratures

1. the midpoint method:

$$
\int_{x_i}^{x_{i+1}} f(x)dx \approx f\left(\frac{x_i + x_{i+1}}{2}\right)h
$$

with the error er $\leq Ch^3$;

2. the Simpson method:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right) h/6$$

with the error er $\leq Ch^5$. Constant $C > 0$ depends on f only.

3. in Matlab we can use the Matlab command

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \text{quad}(f, x_i, x_{i+1})$$

where f is a handle to f.

The unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are obtained as a solution of system (7) and the Rayleigh-Ritz approximation is given by

$$y_n(x) = \sum_{i=1}^{n-1} \alpha_i \phi_i(x).$$

The error analysis for the Galerkin-Ritz method.

It can be shown that

$$|y(x) - y_n(x)| \leq Ch, \quad a \leq x \leq b.$$

for some constant C independent of h. The discussion of this result goes as follows.

The boundary value problem (1) is of our interest. We consider the bilinear form and the load functional related to this problem defined as follows

$$B(v, w) = \int_a^b p(x)v'w' + q(x)vwdx, \quad (f, w) = \int_a^b fwdx$$

for any $v, w \in \mathcal{H}_0^1(a, b)$.
The weak solution u of the problem is defined as folows:

$$B(u, w) = (f, w)$$

for any $w \in \mathcal{H}_0^1(a, b)$.

The variational solution u of the problem is defined as folows::

$$E(u) = \min E(w), \quad \text{for any } w \in \mathcal{H}_0^1(a,b),$$

where

$$E(w) = \frac{1}{2}B(w,w) - (f,w)$$

is the energy functional.

An approximate solution is constructed as follows:

1. $S_h \subseteq \mathcal{H}_0^1(a,b)$ is a chosen finite dimensional subspace
2. $\phi_1(x), \phi_2(x), \ldots, \phi_m(x)$ is a basis in S_h

An approximate solution $u_h \in S_h$ is defined by the relation:

$$B(u_h, \phi) = (f, \phi)$$

for any $\phi \in S_h$ or equivalently (only for the basis functions)

$$B(u_h, \phi_i) = (f, \phi_i), \quad i = 1, 2, \ldots, m.$$

Equivalently, we can define weak solution u_h, as the minimum of the energy functional:

$$E(u_h) = \min E(\phi), \quad \phi(x) \in S_h.$$

We define the error of approximation as the L^2 - norm of the difference $u - u_h$

$$||u - u_h||_{L_2} = ||u - u_h|| = \left(\int_a^b (u(x) - u_h(x))^2 dx \right)^{1/2}.$$

Before we pass to the error analysis of the error, we provide some auxilliary facts.

Auxilliary facts.

We introduce a new norm

$$||v||_B = \left(\int_a^b p(x)v'(x)^2 + q(x)v(x)^2 dx \right)^{1/2} = \sqrt{B(v,v)}$$

Fact 1. There exist constants $\gamma_1, \gamma_2, \gamma_3, \Gamma_1, \Gamma_2 > 0$ such that

a) $\gamma_1 ||v'||_{L_2} \leq ||v||_B \leq \Gamma_1 ||v'||_{L_2}$
b) $\gamma_2 ||v||_{L_2} \leq ||v||_B$
c) $\gamma_3 ||v||_\infty \leq ||v||_B \leq \Gamma_2 ||v'||_B$, where $||v||_\infty = \max |v(x)|$

for any $v \in \mathcal{H}_0^1(a,b)$.

Fact 2. There exists a constant $C > 0$ such that

a) $|B(v, w)| \leq C||v||_B||w||_B$

b) $B(v, v) \geq C||v||_B^2$

Fact 3. Let u be a weak solution and let u_h be an approximate solution. Then

$$B(u - u_h, \phi) = 0$$

for any $\phi \in S_h$.

Proof. It follows from the definition of the weak and approximate solution that

$B(u, \phi) = (f, \phi)$
$B(u_h, \phi) = (f, \phi)$

for any $\phi \in S_h$. Hence we obtain

$$B(u, \phi) - B(u_h, \phi) = B(u - u_h, \phi) = 0.$$

Furthermore, it follows from Fact 2, a) that

$$||u - u_h||_B^2 = B(u - u_h, u - u_h) = B(u - u_h, (u - \phi) + (\phi - u_h)) = B(u - u_h, u - \phi)+$$
$$B(u - u_h, \phi - u_h) = B(u - u_h, u - \phi) \leq C||u - u_h||_B||u - \phi||_B,$$

which gives

The Cea's lemma

$$||u - u_h||_B \leq C||u - \phi||_B, \quad \phi \in S_h$$

or

$$||u - u_h||_B \leq C \min_{\phi} ||u - \phi||_B$$

Example 1. An application of the Cea's lemma. The choice of the S_h space:

Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and

$$S_h = \{\phi(x) \in C[a, b] : \phi(x) \text{ is linear for } x \in [x_{j-1}, x_j],\ j = 1, 2, \ldots, n,\ \phi(a) = \phi(b) = 0.\ \}$$

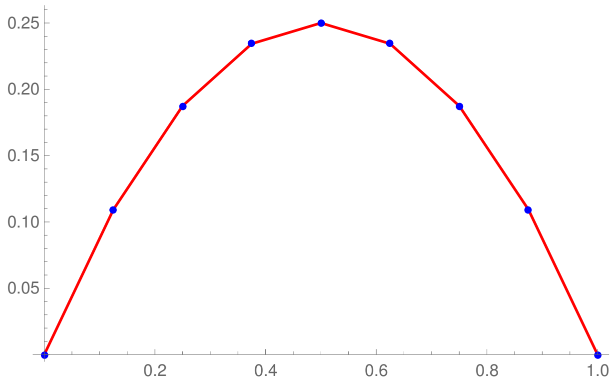


fig. a function $\phi(x) \in S_h$.

The "hat"functions $\phi_j(x)$ form the basis of S_h i.e. $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_{n-1}\phi_{n-1}(x)$.

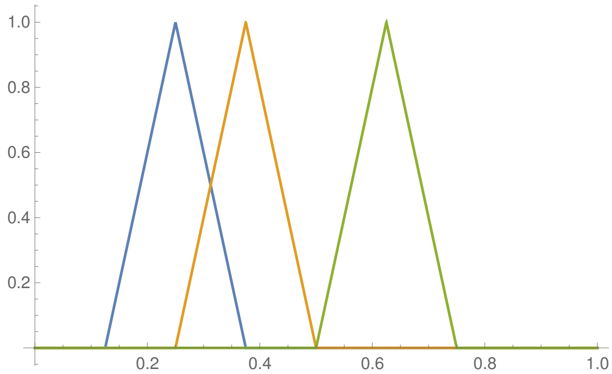
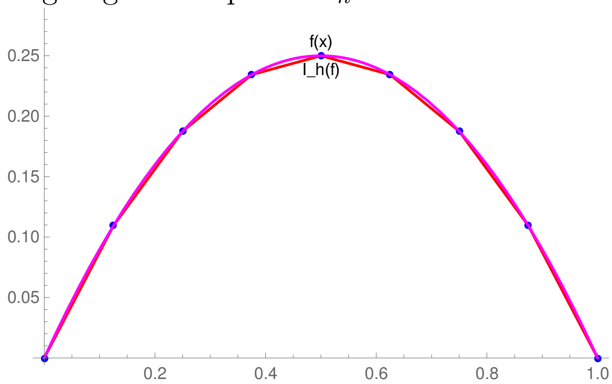


fig. the functions $\phi_i(x)$

Comments. To estimate the error $||u - u_h||_B$ we first consider

$$\min_{\phi \in S_h} ||u - \phi||_B.$$

The latter one can be estimated if one takes a suitably chosen $\phi(x)$, commonly we take the Lagrange's interpolant $I_h u$



It is given by expression

$$I_h u(x) = \sum_{i=1}^{n} u(x_i)\phi_i(x).$$

Due to the Cea's lemma we obtain

$$||u - u_h||_B \leq C \min_{\phi \in S_h} ||u - \phi||_B \leq C||u - I_h u||_B.$$

and further we are concerned with the estimates of $||u - I_h u||_B$.
It is known that there exists a constant $C > 0$ independent of h such that

$$|u(x) - I_h u(x)| \leq Ch^2 \text{ and } |u'(x) - (I_h u)'(x)| \leq Ch$$

for $x \in (x_i, x_{i+1})$ on each interval (x_i, x_{i+1}).

In practice, we estimate the error in L^2 norm:

$$\text{error}_2(h) = ||u - u_h||_2 = \left(\int_a^b (u(x) - u_h(x))^2 dx \right)^{1/2}$$

and in the H^1 norm:

$$\text{error}_{H^1}(h) = (||u - u_h||_2^2 + ||u' - u_h'||_2^2)^{1/2}$$

equivalent to the norm $||u - u_h||_B$.

Calculations, for approximations of integrals we use the Simpson method:

1. $||u - u_h||_2^2 = \int_a^b (u - u_h)^2 dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (u - u_h)^2 dx$
2. $\int_{x_i}^{x_{i+1}} (u - u_h)^2 dx = \int_{x_i}^{x_{i+1}} \left(u(x) - \left(\frac{u_{h,i+1} - u_{h,i}}{h} (x - x_i) + u_{h,i} \right) \right)^2 dx \approx$

$$((u(x_i) - u_{h,i})^2 + 4 \left(u(x_{i+\frac{1}{2}}) - \frac{u_{h,i+1} + u_{h,i}}{2} \right)^2 + (u(x_{i+1}) - u_{h,i+1})^2) \frac{h}{6}$$

where $u_{h,i} = u_h(x_i)$, $x_{i+\frac{1}{2}} = 0.5(x_i + x_{i+1})$.

1. $||u' - u_h'||_2^2 = \int\limits_a^b (u' - u_h')^2 dx = \sum\limits_{i=0}^{n-1} \int\limits_{x_i}^{x_{i+1}} (u' - u_h')^2 dx$

2. $\int\limits_{x_i}^{x_{i+1}} (u' - u_h')^2 dx = \int\limits_{x_i}^{x_{i+1}} \left(u'(x) - \frac{u_{h,i+1} - u_{h,i}}{h} \right)^2 dx \approx$

$$\left(\left(u'(x_i) - \frac{u_{h,i+1} - u_{h,i}}{h} \right)^2 + 4 \left(u'(x_{i+\frac{1}{2}}) - \frac{u_{h,i+1} - u_{h,i}}{h} \right)^2 + \left(u'(x_{i+1}) - \frac{u_{h,i+1} - u_{h,i}}{h} \right)^2 \right) \frac{h}{6}$$

We expect the following convergence rates

$$\text{error}_2(h) \leq Ch^2, \;\; \text{error}_{H^1}(h) \leq Ch$$