Lecture 4

The finite difference methods for the hyperbolic differential equations (the wave equations).

Let us consider the following problem

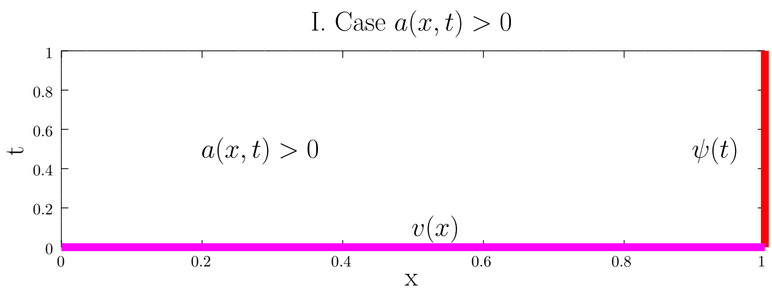
$$(1)\quad \frac{\partial u}{\partial t} = a(x,t)\frac{\partial u}{\partial x} + f(x,t), \quad x \in (\alpha, \beta), \quad t \in (0,T),$$

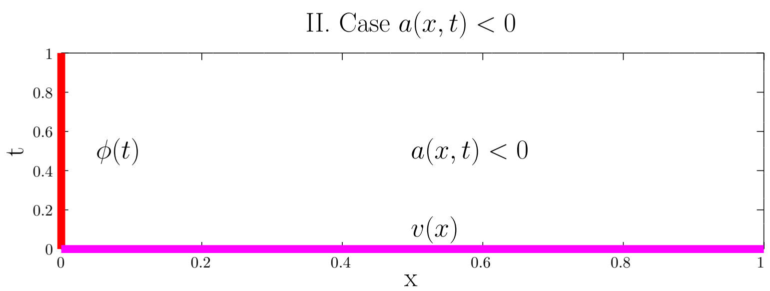
where the coefficient $a = a(x,t)$ may be nonconstant but it is assumed to be still either positive, $a(x,t) > 0$ or negative, $a(x,t) < 0$ for all (x,t), i.e. $a(x,t)$ has a constant sign. The problem is completed with initial data

$$(2)\quad u(x,0) = v(x), \quad x \in (\alpha, \beta),$$

and depending on the sign of $a(x,t)$ with the boundary condition

$$(3)\quad \begin{cases} u(\alpha,t) = \phi(t), & \text{if } a(x,t) < 0 \\ u(\beta,t) = \psi(t), & \text{if } a(x,t) > 0 \end{cases}$$





Remark. In the case when $a(x,t)$ changes its sign inside the region $x \in (\alpha, \beta)$, $t \in (0,T)$ the problem becomes more complicated and solutions may have singularities.

Example. Let

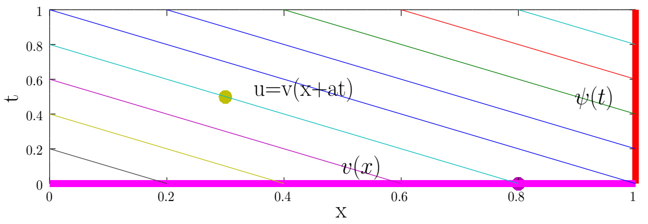
$$u_t = au_x, \text{ where } a \text{ is constant}$$
$$v(x) = u(x,0).$$

We check that the solution of this equation is given by $u(x,t) = v(x + at)$. We see that a special role is played by lines $x + at = \text{const}$, in this case it follows from this formula, that $u(x,t)$ is constant along those. Those lines are called the characteristics of the equation. We find them in a parametric form $(x = x(s), y = y(s))$ as the solutions of the following problem

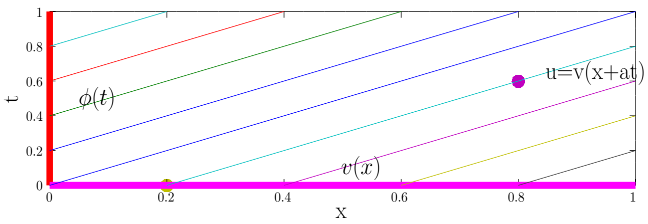
$$\begin{cases} t'(s) = 1 \\ x'(s) = -a \end{cases}$$

whose solution is equal $(t = s + t(0), x = -as + x(0))$, usually we take $t(0) = 0$, hence $(t = s, x + at = x(0)(= \text{const}))$

I. Characteristics x+at=const, for $u_t = au_{xx}$, case $a > 0$



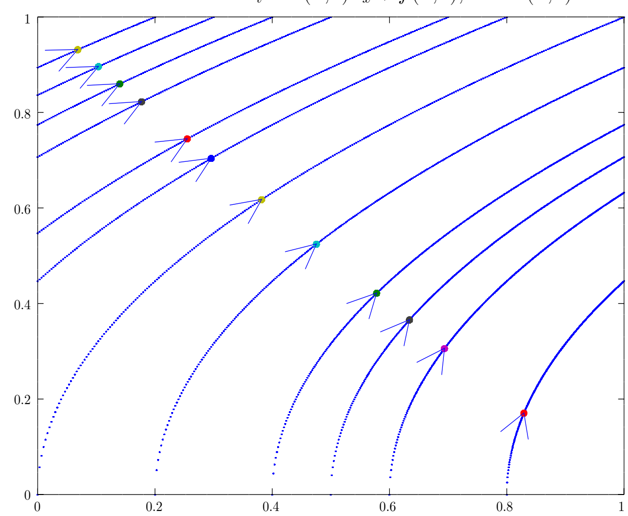
II. Characteristics x+at=const, case $a < 0$



Remark. A similar statement holds for variable coefficient $a = a(x,t)$, in which case the characteristic is curved and its parametric form $(t = t(s), x = x(s))$ is obtained as a solution of the system of the differential equations

$$\begin{cases} t'(s) = 1 \\ x'(s) = -a(x,t) \\ t(0) = 0, \; x(0) = x_0 \end{cases}$$

1. Characteristics for $u_t = a(x,t)u_x + f(x,t)$, case $a(x,t) < 0$

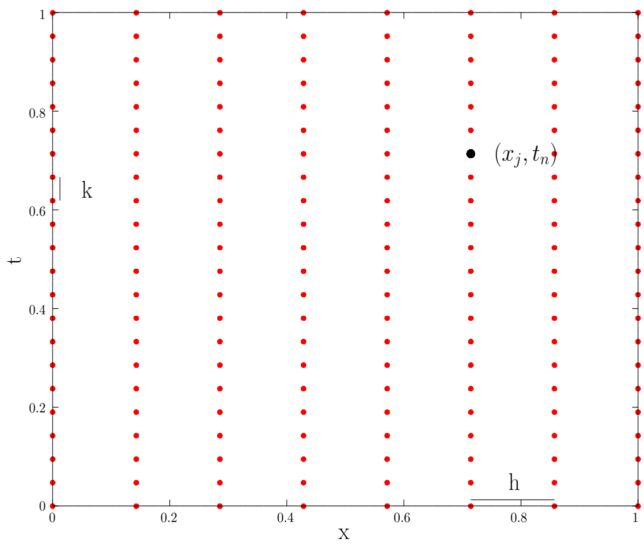


Discretization of the problem.

We discretize the intervals
$\alpha = x_0 < x_1 < \cdots < x_n = \beta$, $h = (\beta - \alpha)/n$
$0 = t_0 < t_1 < \cdots < t_m = T$, $k = (T - 0)/m$.

We define the approximate solution as a number sequence $\{u_{ij}\}$ such that $u_{i0} = v(x_i)$, $i = 0, 1, \ldots, n$ $u_{0j} = \phi(t_j)$, $j = 0, 1, 2, \ldots, m$ and $u_{ij} \approx u(x_i, t_j)$ $i = 1, 2, 3, \ldots, n$,

$j = 1, 2, \ldots, m$.



There are many schemes for constructing sequence $\{u_{ij}\}$. We consider the following ones

1. the forward explicit scheme

2. the backward implicit scheme

3. the Crank-Nicolson scheme

We begin with a consideration of

1. the forward explicit scheme

Case $a(x,t) > 0$.

$$\frac{u_{i,j+1} - u_{ij}}{k} = a_{ij}\frac{u_{i+1,j} - u_{ij}}{h} + f_{ij}, \quad i = 0, 1, 2, \ldots, n-1, \; j = 0, 1, 2, \ldots, m-1, \tag{4}$$

where $a_{ij} = a(x_i, t_j)$. Hence we obtain

$$\begin{aligned} &u_{i,j+1} = \lambda a_{i,j} u_{i+1,j} + (1 - \lambda a_{i,j}) u_{ij} + k f_{ij}, \quad i = 0, 1, 2, \ldots, n-1, \; j = 0, 1, 2, \ldots, m-1, \\ &u_{i,0} = v_i, \quad u(n,j) = \psi_j, \quad i = 0, 1, 2, \ldots, n-1, \; j = 0, 1, 2, \ldots, m-1, \end{aligned} \tag{5}$$

where $\lambda = k/h$, $v_i = v(x_i)$, $\psi_j = \psi(t_j)$. In the matrix form we have

$$\begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 1 - \lambda a_{0,j} & \lambda a_{0,j} & 0 & \ldots & 0 \\ 0 & 1 - \lambda a_{1,j} & \lambda a_{1,j} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 - \lambda a_{n-1,j} \end{bmatrix} \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} + \tag{6}$$

$$\lambda a_{n-1,j} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u_{n,j} \end{bmatrix} + k \begin{bmatrix} f_{0,j} \\ f_{1,j} \\ \vdots \\ f_{n-1,j} \end{bmatrix}$$

in a short form

$$(7) \quad \vec{u}_{j+1} = A\vec{u}_j + \lambda a_{n-1,j}\vec{w}_j + k\vec{f}_j$$

Starting with the vector $\vec{u}_0 = \begin{bmatrix} u_{0,0} \\ u_{1,0} \\ \vdots \\ u_{n-1,0} \end{bmatrix}$ and the value $u_{n,0}$ we get subsequently the vectors of the approximate values $\vec{u}_1 = \begin{bmatrix} u_{0,1} \\ u_{1,1} \\ \vdots \\ u_{n-1,1} \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ \vdots \\ u_{n-1,2} \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} u_{0,3} \\ u_{1,3} \\ \vdots \\ u_{n-1,3} \end{bmatrix}$, ...

The stability condition of the method = the CFL condition (Courant-Friedrichs-Levy condition):

$$(8) \quad \lambda \max_{ij} |a_{ij}| \leq 1 \ \ \text{(the CFL condition)}$$

Case $a(x,t) < 0$.

$$(9) \quad \frac{u_{i,j+1} - u_{ij}}{k} = a_{ij}\frac{u_{i,j} - u_{i-1,j}}{h} + f_{ij}, \ \ i = 1, 2, \dots, n, \ j = 0, 1, 2, \dots, m-1,$$

Hence we obtain

$$\begin{aligned} & u_{i,j+1} = -\lambda a_{ij}u_{i-1,j} + (1 + \lambda a_{ij})u_{ij} + kf_{ij}, \ \ i = 1, 2, \dots, n, \ j = 0, 1, 2, \dots, m-1, \\ (10) \quad & u_{i,0} = v_i, \ \ u(0,j) = \phi_j, \ \ i = 1, 2, \dots, n, \ j = 0, 1, 2, \dots, m-1, \end{aligned}$$

where $\lambda = k/h$, $v_i = v(x_i)$, $\phi_j = \phi(t_j)$. In the matrix form we have

$$(11) \quad \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} 1+\lambda a_{1,j} & 0 & 0 & \dots & 0 \\ -\lambda a_{2,j} & 1+\lambda a_{2,j} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda a_{n,j} & 1+\lambda a_{n,j} \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} - \lambda a_{1,j} \begin{bmatrix} u_{0,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + k \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{n,j} \end{bmatrix}.$$

The stability method - the CFL condition (8).

2. the backward implicit scheme

Case $a(x,t) > 0$.

$$(12)\quad \frac{u_{i,j+1}-u_{i,j}}{k} = a_{i,j+1}\frac{u_{i+1,j+1}-u_{i,j+1}}{h} + f_{i,j+1},\ \ i=0,1,2,\dots,n-1,\ j=0,1,2,\dots,m-1,$$

where $a_{ij} = a(x_i, t_j)$. Hence we obtain

$$(13)\quad \begin{aligned} &-\lambda a_{i,j+1}u_{i+1,j+1} + (1+\lambda a_{i,j+1})u_{i,j+1} = u_{i,j} + kf_{i,j+1},\\ &i=0,1,2,\dots,n-1,\ j=0,1,2,\dots,m-1,\\ &u_{i,0} = v_i,\ \ u(n,j) = \psi_j,\ \ i=0,1,2,\dots,n-1,\ j=0,1,2,\dots,m-1, \end{aligned}$$

where $\lambda = k/h$, $v_i = v(x_i)$, $\psi_j = \psi(t_j)$. In the matrix form we have

$$(14)\quad \begin{bmatrix} 1+\lambda a_{0,j+1} & -\lambda a_{0,j+1} & 0 & \dots & 0 \\ 0 & 1+\lambda a_{1,j+1} & -\lambda a_{1,j+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1+\lambda a_{n-1,j+1} \end{bmatrix} \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} =$$
$$\begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} + \lambda a_{n-1,j+1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u_{n,j+1} \end{bmatrix} + k \begin{bmatrix} f_{0,j+1} \\ f_{1,j+1} \\ \vdots \\ f_{n-1,j+1} \end{bmatrix}$$

1. Since the determinant $\det(A) = (1+\lambda a_{0,j+1})(1+\lambda a_{1,j+1})\dots(1+\lambda a_{n-1,j+1}) \neq 0$, the system (19) has allways a unique solution. Starting with the vector $\begin{bmatrix} u_{0,0} \\ u_{1,0} \\ \vdots \\ u_{n-1,0} \end{bmatrix}$ and the value $u_{n,1}$ we solve subsequently the system of equations (19) which gives the vectors of the approximate values $\begin{bmatrix} u_{0,1} \\ u_{1,1} \\ \vdots \\ u_{n-1,1} \end{bmatrix}, \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ \vdots \\ u_{n-1,2} \end{bmatrix}, \begin{bmatrix} u_{0,3} \\ u_{1,3} \\ \vdots \\ u_{n-1,3} \end{bmatrix}, \dots.$

2. The method is stable for $h, k > 0$.

3. The method is of the first order accurate.

Case $a(x,t) < 0$.

$$(15)\quad \frac{u_{i,j+1}-u_{i,j}}{k} = a_{i,j+1}\frac{u_{i,j+1}-u_{i-1,j+1}}{h} + f_{i,j+1},\ \ i=1,2,\dots,n,\ j=0,1,2,\dots,m-1,$$

Hence we obtain

$$(16)\quad \begin{aligned} &\lambda a_{i,j+1}u_{i-1,j+1} + (1-\lambda a_{i,j+1})u_{i,j+1} = u_{i,j} + kf_{i,j+1},\\ &i=1,2,\dots,n,\ j=0,1,2,\dots,m-1,\\ &u_{i,0} = v_i,\ \ u(0,j) = \phi_j,\ \ i=1,2,\dots,n,\ j=0,1,2,\dots,m, \end{aligned}$$

where $\lambda = k/h$, $v_i = v(x_i)$, $\phi_j = \phi(t_j)$. In the matrix form we have

$$(17)\quad \begin{bmatrix} 1-\lambda a_{1,j+1} & 0 & 0 & \dots & 0 \\ \lambda a_{2,j+1} & 1-\lambda a_{2,j+1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda a_{n,j+1} & 1-\lambda a_{n,j+1} \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} =$$
$$\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} - \lambda a_{1,j+1} \begin{bmatrix} u_{0,j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + k \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ \vdots \\ f_{n,j+1} \end{bmatrix}.$$

1. Since the determinant $\det(A) = (1-\lambda a_{1,j+1})(1-\lambda a_{2,j+1})\dots(1-\lambda a_{n,j+1}) \neq 0$, the system (21) has alweays a unique solution. Starting with the vector $\begin{bmatrix} u_{0,0} \\ u_{1,0} \\ \vdots \\ u_{n-1,0} \end{bmatrix}$ and the value $u_{0,1}$ and solving subsequently the system of equations (21) we get the vectors of the approximate values $\begin{bmatrix} u_{0,1} \\ u_{1,1} \\ \vdots \\ u_{n-1,1} \end{bmatrix}, \begin{bmatrix} u_{0,2} \\ u_{1,2} \\ \vdots \\ u_{n-1,2} \end{bmatrix}, \begin{bmatrix} u_{0,3} \\ u_{1,3} \\ \vdots \\ u_{n-1,3} \end{bmatrix}, \dots$.

2. The method is stable for $h, k > 0$.

3. The method is of the first order accurate.

3. The Crank-Nicolson scheme

the case $a(x,t) > 0$

The basic relation

$$(18)\quad \frac{u_{i,j+1}-u_{i,j}}{k} = \frac{1}{2}a_{i,j+1}\frac{u_{i+1,j+1}-u_{i,j+1}}{h} + \frac{1}{2}f_{i,j+1} + \frac{1}{2}a_{ij}\frac{u_{i+1,j}-u_{ij}}{h} + \frac{1}{2}f_{ij},$$
$$i = 0,1,2,\dots,n-1,\ j = 0,1,2,\dots,m-1,$$

The matrix form of the algorithm

$$
(19)\quad \begin{bmatrix} 1+\frac{1}{2}\lambda a_{0,j+1} & -\frac{1}{2}\lambda a_{0,j+1} & 0 & \dots & 0 \\ 0 & 1+\frac{1}{2}\lambda a_{1,j+1} & -\frac{1}{2}\lambda a_{1,j+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1+\frac{1}{2}\lambda a_{n-1,j+1} \end{bmatrix} \begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} =
$$

$$
\begin{bmatrix} 1-\frac{1}{2}\lambda a_{0,j} & \frac{1}{2}\lambda a_{0,j} & 0 & \dots & 0 \\ 0 & 1-\frac{1}{2}\lambda a_{1,j} & \frac{1}{2}\lambda a_{1,j} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1-\frac{1}{2}\lambda a_{n-1,j} \end{bmatrix} \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} +
$$

$$
\frac{1}{2}\lambda a_{n-1,j+\frac{1}{2}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u_{n,j+\frac{1}{2}} \end{bmatrix} + k \begin{bmatrix} f_{0,j+\frac{1}{2}} \\ f_{1,j+\frac{1}{2}} \\ \vdots \\ f_{n-1,j+\frac{1}{2}} \end{bmatrix}
$$

where $f_{i,j+\frac{1}{2}} = \frac{1}{2}(f_{ij} + f_{ij+1})$, $a_{n-1,j+\frac{1}{2}} = \frac{1}{2}(a_{n-1,j} + a_{n-1,j+1})$

the case $a(x,t) < 0$

The basic relations

$$
(20)\quad \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}a_{i,j+1}\frac{u_{i,j+1} - u_{i-1,j+1}}{h} + \frac{1}{2}f_{i,j+1} + \frac{1}{2}a_{ij}\frac{u_{i,j} - u_{i-1,j}}{h} + \frac{1}{2}f_{ij},
$$
$$
i = 1, 2, \dots, n,\ j = 0, 1, 2, \dots, m-1,
$$

The matrix form of the algorithm

$$
(21)\quad \begin{bmatrix} 1-\frac{1}{2}\lambda a_{1,j+1} & 0 & 0 & \dots & 0 \\ \frac{1}{2}\lambda a_{2,j+1} & 1-\frac{1}{2}\lambda a_{2,j+1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{2}\lambda a_{n,j+1} & 1-\frac{1}{2}\lambda a_{n,j+1} \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} =
$$

$$
\begin{bmatrix} 1+\frac{1}{2}\lambda a_{1,j} & 0 & 0 & \dots & 0 \\ -\frac{1}{2}\lambda a_{2,j} & 1+\frac{1}{2}\lambda a_{2,j} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\frac{1}{2}\lambda a_{n,j} & 1+\frac{1}{2}\lambda a_{n,j} \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} -
$$

$$
\lambda a_{1,j+\frac{1}{2}} \begin{bmatrix} u_{0,j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + k \begin{bmatrix} f_{1,j+\frac{1}{2}} \\ f_{2,j+\frac{1}{2}} \\ \vdots \\ f_{n,j+\frac{1}{2}} \end{bmatrix}.
$$