The coefficients $a_k(0)$ are retrieved from the initial condition as follows

$$v(x) = \sum_{k=1}^{\infty} v_k \sin\left(k\pi \frac{x}{l}\right),$$

$$(10)\quad v_k = \int_0^l v(x) \sin\left(k\pi \frac{x}{l}\right) dx \Big/ \int_0^l \sin^2\left(k\pi \frac{x}{l}\right) dx = \frac{2}{l} \int_0^l v(x) \sin\left(k\pi \frac{x}{l}\right) dx,$$

$$a_k(0) = v_k, \quad k = 1, 2, 3, \ldots$$

Remark

In practise, the integral in (30) is calculated approximately by using for example the complex Simpson method

$$(11)\quad \int_0^l f(s) ds \approx \frac{1}{6} h \sum_{i=0}^{n} \Big(f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1}) \Big)$$

where $0 = x_0 \le x_1 \le x_2 \le \ldots x_{n+1} = l$, $h = (x_{i+1} - x_i)/n$, $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$.

The finite difference methods for the hyperbolic differential equations.

Let us consider the wave equation as the model problem.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), & 0 < x < l, \quad t > 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = h(x), & 0 \le x \le l, \\ \frac{\partial u}{\partial t}(x,0) = g(x), & 0 \le x \le l, \end{cases}$$

where $f(x)$, $g(x)$, $h(x)$ are given smooth bounded functions.

1. The forward explicit Euler's scheme produces the sequence of approximate values for the solution by relations

$$\frac{U_i^{n-1} - 2U_i^n + U_i^{n+1}}{k^2} = \alpha^2 \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2} + f_i^n,$$

where $f_i^n = f(x_i, t_n)$. If $\lambda = \alpha \frac{k}{h}$, we can write this difference equation in the vector form

$$(12)\quad IU^{n-1} - 2IU^n + IU^{n+1} = \lambda^2 AU^n + k^2 f^n$$

where $U^n = [U_1^n, U_2^n, \dots, U_{m-1}^n]'$, similarly U^{n-1} and U^{n+1},
$f^n = [f_1^n, f_2^n, \dots, f_{m-1}^n]'$,

$$(13)\quad A = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and I is identity matrix of the size $(m-1) \times (m-1)$. After a modification

$$(14)\quad U^{n+1} = (2I + \lambda^2 A)U^n - IU^{n-1} + k^2 f^n.$$

This equation holds for each $n = 1, 2, \dots$. The boundary conditions give

$$(15)\quad U_0^n = U_m^n = 0,$$

for each $n = 1, 2, 3, \dots$, and the initial condition implies that

$$(16)\quad U_i^0 = h(x_i)$$

for each $i = 1, 2, \dots, m-1$.
Writing (14) in a matrix form we obtain

$$(17)\quad \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & 0 & \dots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{m-1}^n \end{bmatrix} -$$

$$\begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2 \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{m-1}^n \end{bmatrix}$$

Equations (14) (or (17)) imply that the $(n+1)$-st time step requires values from n-th and $(n-1)$-st time steps. This produces a minor starting problem since the values for $n = 0$ are given by equation (16), but the values for $n = 1$, which are needed in equation (14) to compute U_i^2 must be obtained from the initial velocity condition

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 \le x \le l.$$

The first approach is to replace $(\partial u/\partial t)$ by a forward-difference approximation, which results in

$$(18)\quad U_i^1 = U_i^0 + kg(x_i)$$

A better approximation to $u(x_i, t_1)$ can be rather easily obtained particularly when the second derivative of f at x_i can be determined.

$$(19)\quad U_i^1 = U_i^0 + kg(x_i) + \frac{\alpha^2 k^2}{2} h''(x_i).$$

This is an approximation with the local truncation error $O(k^2)$ for each $i = 1, 2, \ldots, m-1$.
If $h \in C^4[0,1]$ but $h''(x_i)$ is not readily available we can use an approximation

$$(20)\quad U_i^1 = (1-\lambda^2)h(x_i) + \frac{\lambda^2}{2}h(x_{i+1}) + \frac{\lambda^2}{2}h(x_{i-1}) + kg(x_i),$$

to approximate U_i^1 for each $i = 1, 2, \ldots, m-1$.

2. The backward implicit Euler's scheme produces the sequence of approximate values for the solution by relations

$$\frac{U_i^{n-1} - 2U_i^n + U_i^{n+1}}{k^2} = \alpha^2 \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2} + f_i^{n+1}$$

If $\lambda = \alpha\frac{k}{h}$, we can write the difference equation as

$$U^{n-1} - 2U^n + U^{n+1} = \lambda^2 AU^{n+1} + k^2 f^{n+1}$$

or

$$(21)\quad (I - \lambda^2 A)U^{n+1} = 2U^n - U^{n-1} + k^2 f^{n+1}.$$

This equation holds for each $n = 1, 2, \ldots$. The boundary conditions give

$$U_0^n = U_m^n = 0,$$

for each $n = 1, 2, 3, \ldots$, and the initial condition implies that

$$U_i^0 = h(x_i)$$

for each $i = 1, 2, \ldots, m-1$. Vector $[U_1^1, U_2^1, \ldots, U_{m-1}^1]$ is calculated using one of the formula (18), (19) or (20).
Writing relation (21) in a matrix form we obtain

$$(22)\quad B\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = 2\begin{bmatrix} U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{m-1}^{n} \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2\begin{bmatrix} f_1^{n+1} \\ f_2^{n+2} \\ \vdots \\ f_{m-1}^{n+1} \end{bmatrix}$$

where

$$B = \begin{bmatrix} 1+2\lambda^2 & -\lambda^2 & 0 & 0 & \dots & 0 \\ -\lambda^2 & 1+2\lambda^2 & -\lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -\lambda^2 & 1+2\lambda^2 \end{bmatrix}$$

3. The Cranck-Nicolson scheme, as the stable second order accurate method. It can be obtained informally in the following way

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + f_j^n$$
$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} + f_j^{n+1}$$

Taking the average of the above equalities we obtain

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{1}{2}\left(\frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2}\right) + f_j^{n+\frac{1}{2}},$$

where $f_j^{n+\frac{1}{2}} = \frac{1}{2}(f_j^n + f_j^{n+1})$.

After some simple transformations we obtain

$$U^{n+1} - 2U^n + U^{n-1} = \frac{1}{2}\lambda^2 AU^n + \frac{1}{2}\lambda^2 AU^{n+1} + k^2 f^{n+\frac{1}{2}}$$

or finally

$$(23)\quad \left(I - \frac{1}{2}\lambda^2 A\right) U^{n+1} = \left(2I + \frac{1}{2}\lambda^2 A\right) U^n - U^{n-1} + k^2 f^{n+\frac{1}{2}}$$

$$U_0^{n+1} = U_m^{n+1} = 0,$$

where A is the earlier introduced matrix in (13). Introducing the matrices

$$B = I - \frac{1}{2}\lambda^2 A = \begin{bmatrix} 1+\lambda^2 & -\frac{1}{2}\lambda^2 & 0 & 0 & \dots & 0 \\ -\frac{1}{2}\lambda^2 & 1+\lambda^2 & -\frac{1}{2}\lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{2}\lambda^2 & 1+\lambda^2 \end{bmatrix}$$

$$C = 2I + \frac{1}{2}\lambda^2 A = \begin{bmatrix} 2-\lambda^2 & \frac{1}{2}\lambda^2 & 0 & 0 & \dots & 0 \\ \frac{1}{2}\lambda^2 & 2-\lambda^2 & \frac{1}{2}\lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2}\lambda^2 & 2-\lambda^2 \end{bmatrix}$$

the Cranck-Nicolson scheme can be written in the matrix form

$$B \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = C \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{m-1}^{n} \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2 \begin{bmatrix} f_1^{n+\frac{1}{2}} \\ f_2^{n+\frac{1}{2}} \\ \vdots \\ f_{m-1}^{n+\frac{1}{2}} \end{bmatrix}$$

4. The Fourier method

1. We are looking for a solution of the form $u(x,t) = \sum_{k=1}^{\infty} a_k(t)\sin(k\pi\frac{x}{l})$ If this series is sufficiently fast convergent, then the function $u(x,t)$ can be considered as the solution of the problem. In order to determine the coefficients $a_k(t)$ we make the observations

(24) $$u_{tt}(x,t) = \sum_{k=1}^{\infty} a_k''(t)\sin(k\pi\frac{x}{l})$$

(25) $$u_{xx}(x,t) = \sum_{k=1}^{\infty} -\frac{k^2\pi^2}{l^2} a_k(t)\sin(k\pi\frac{x}{l})$$

(26) $$f(x,t) = \sum_{k=1}^{\infty} f_k(t)\sin(k\pi\frac{x}{l})$$

where coefficients $f_k(t)$ are obtained from the general formulas for the Fourier expansion series

(27) $$f_k(t) = \int_0^l f(x,t)\sin(k\pi\frac{x}{l})dx / \int_0^l \sin^2(k\pi\frac{x}{l})dx = \frac{2}{l}\int_0^l f(x,t)\sin(k\pi\frac{x}{l})dx$$

Now from the differential equation it follows that

$$\sum_{k=1}^{\infty} a_k''(t)\sin(k\pi\frac{x}{l}) =$$
$$\alpha^2\sum_{k=1}^{\infty} -\frac{k^2\pi^2}{l^2}a_k(t)\sin(k\pi\frac{x}{l}) + \sum_{k=1}^{\infty} f_k(t)\sin(k\pi\frac{x}{l})$$

Hence we obtain equations

$$(28)\quad a_k''(t) = -\frac{k^2\alpha^2\pi^2}{l^2}a_k(t) + f_k(t), \quad k = 1, 2, \ldots$$

which have the following solutions

$$(29)\quad a_k(t) = a_k(0)\cos\left(\frac{k\alpha\pi}{l}t\right) + \frac{l}{k\alpha\pi}a_k'(0)\sin\left(\frac{k\alpha\pi}{l}t\right) +$$
$$\frac{l}{k\alpha\pi}\int_0^t f_k(s)\sin\left(\frac{k\alpha\pi}{l}(t-s)\right)ds, \quad k = 1, 2, \ldots$$

The coefficients $a_k(0)$ are retrieved from the initial condition as follows

$$h(x) = \sum_{k=1}^{\infty} h_k \sin\left(k\pi\frac{x}{l}\right),$$

where

$$(30)\quad h_k = \int_0^l h(x)\sin\left(k\pi\frac{x}{l}\right)dx \Big/ \int_0^l \sin^2\left(k\pi\frac{x}{l}\right)dx = \frac{2}{l}\int_0^l h(x)\sin\left(k\pi\frac{x}{l}\right)dx,$$

which gives

$$a_k(0) = h_k, \quad k = 1, 2, 3, \ldots$$

the coefficients $a_k'(0)$ are retrieved from the initial condition as follows

$$g(x) = \sum_{k=1}^{\infty} g_k \sin\left(k\pi\frac{x}{l}\right),$$

where

$$g_k = \int_0^l g(x) \sin\left(k\pi \frac{x}{l}\right) dx \Big/ \int_0^l \sin^2\left(k\pi \frac{x}{l}\right) dx = \frac{2}{l} \int_0^l g(x) \sin\left(k\pi \frac{x}{l}\right) dx, \tag{31}$$

which gives

$$a_k'(0) = g_k, \quad k = 1, 2, 3, \ldots$$

Remark

In practise, the integrals in (27), (29), (30) and (31) are calculated approximately by using for example the complex Simpson method

$$\int_0^l f(s)ds \approx \frac{1}{6} h \sum_{i=0}^{n} \Big(f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1}) \Big) \tag{32}$$

where $0 = x_0 \leq x_1 \leq x_2 \leq \ldots x_{n+1} = l$, $h = (x_{i+1} - x_i)/n$, $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$.