Lecture 1

I. The initial value problem for ordinary differential equations

$$\begin{cases} y' = f(t,y), & t \in (a,b) \\ y(t_0) = y_0, & t_0 \in (a,b) \end{cases}$$

The most important questions are

1. Existence and uniqueness of solutions to the problem.
2. Continuous dependence on the data.
3. The interval of the existence of the solution.

The answers to the question 1 and 2: if f is regular, i. e. $f \in C^1([a,b] \times R)$, then there exists exactly one solution of the problem and it is continuous with respect to the data.

The explanation of 2:

Let's consider the original and perturbed problems

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \qquad \begin{cases} y' = f(t,y,\epsilon) \\ y(t_0) = y_{\epsilon,0}, \quad y_\epsilon = y(t,\epsilon) \end{cases}$$

Then a continuous dependence means that

$$\forall \delta > 0 \; \exists \epsilon_0 > 0 \; \forall y_{0,\epsilon} \; \forall \; 0 < \epsilon < \epsilon_0 \;\; (|y_0 - y_{\epsilon,0}| < \epsilon_0) \Rightarrow |y(t) - y_\epsilon(t)| < \delta, \;\; t \in (a,b)$$

The remark to the question 3:

Example 1. Let's consider the following problem

$$\begin{cases} y' = y^2 \\ y(0) = c \end{cases}$$

here $f(t,y) = y^2$ and $y(t) = \frac{c}{1-ct}, \;\; t \in (0, 1/c)$ is a solution. We see that the solution exists on the interval, whose length is not fixed but strongly depends on initial data c. Generally, determining the length of the interval on which solution exists is a very difficult task.

Remark. 1. There are no separate numerical methods for systems of the differential equations. All the above considerations concern also to the systems.

$$\begin{cases} y_1' = f_1(t, y_1, y_2, \ldots, y_m) \\ y_2' = f_2(t, y_1, y_2, \ldots, y_m) \\ \ldots \\ y_m' = f_m(t, y_1, y_2, \ldots, y_m) \\ \ \ y(t_0) = y^0, \end{cases}$$

which we write in the vector form differential equation

$$y'(t) = f(t, y), \ \ y(t_0) = y^0,$$

where

$$y(t) = (y_1(t), y_2(t), \ldots, y_m(t)), \ \ y^0 = (y_1^0, y_2^0, \ldots, y_m^0)$$
$$f(t, y) = (f_1(t, y_1, y_2, \ldots, y_m), f_2(t, y_1, y_2, \ldots, y_m) \ldots, f_m(t, y_1, y_2, \ldots, y_m))$$

Example 2. There are no separate numerical methods for higher order differential equations. In such a case we transform the equation of higher order to the system of the first order differential equation, as in the following example.

$$y'' = f(t, y, y'), \ \ y(t_0) = y_1, \ \ y'(t_0) = y_2,$$

We introduce $u_1(t) = y(t), \ \ u_2(t) = y'(t), \ \ u_1^0 = y_1, \ \ u_2^0 = y_2$. Then we get the system of the first order differential equations

$$\begin{cases} u_1' = u_2, \\ u_2' = f(t, u_1, u_2), \ \ u(t_0) = u^0 \end{cases}$$

II. Approximate solution

1. Discretization of the problem.

We begin with a division: $a = t_0 < t_1 < \cdots < t_n = b$, sequence $\{t_i\}$ is called a grid, mesh points, knots or nodes. We usually use the equispaced nodes (a uniform grid), where $h = t_{i+1} - t_i$ for every $0 \leq i \leq n-1$ but this is not a rule. The parameter h is called a grid size or a step size of the method.

The aim of any approximate method is a construction of a sequence of the values $\{y_i\}$ which can be considered as approximations to the exact values $\{y(t_i)\}$ of the solution, $y_i \approx y(t_i)$.

2. The simplest constructions of y_i are the following

a. the explicit Euler method

$$y_0 = y(t_0), \ \ y_{i+1} = y_i + hf(t_i, y_i).$$

b. the implicit Euler method

$$y_0 = y(t_0), \ \ y_{i+1} = y_i + hf(t_{i+1}, y_{i+1}).$$

To provide motivation for the form of the explicit method, let us consider the sequence of exact values $\{y(t_i)\}$. We observe that

$$y(t_{i+1}) - y(t_i) - hf(t_i, y(t_i)) = y(t_i + h) - y(t_i) - hy'(t_i) = r(h),$$

where $r(h)$ is the remainder in the Taylor formula, it can be expressed as $r(h) = \frac{1}{2}y''(\xi)h^2$. Hence, it follows that there exists a constant $c > 0$ such that $|r(h)| \leq ch^2$. Since the parameter h is small ($h \to 0$), the remainder $r(h)$ is also small. Finally we get a conclusion: the sequence $\{y_i(t)\}$ almost exactly satisfies the explicit Euler algorithm
$y_{i+1} - y_i - hf(t_i, y_i) = 0$, the error, $r(h)$ is small. Whence we expect that terms y_i are close to the respective terms $y(t_i)$, $i = 1, 2, \dots, n$.

More general discrete schemes :

$$y_{k+1} = a_{m-1}y_k + a_{m-2}y_{k-1} + \dots + a_0 y_{k+1-m} + hF(t_k, h, y_{k+1}, y_k, \dots, y_{k+1-m})$$

with given initial data $y_0, y_1, \dots, y_{m-1}$.

Such schemes are called the difference schemes or m-step methods.

Explicit and implicit methods:

When y_{k+1} is absent on the right hand side, the scheme is called explicit, otherwise it is implicit. Let us note that the explicit and implicit Euler methods are one-step methods.

2. Basic properties of approximate algorithms

2a. The most important property of the numerical method is its **consistency** with the differential problem.

Let's consider the exact values $y_i(t)$ and their approximations y_i. Since $y_i \approx y(t_i)$, we expect that for the local error of the method $e_l^k(h)$ we have

$$\begin{aligned} e_l^k(h) = y(t_{k+1}) - a_{m-1}y(t_k) - a_{m-2}y(t_{k-1}) - \dots - a_0 y(t_{k+1-m}) \\ -hF(t_k, h, y(t_{k+1}), y(t_k), \dots, y(t_{k+1-m})) \approx 0 \end{aligned}$$

(compare the definition of y_i.)
We say that the difference scheme is consistent with the differential equation, when

$$\max_k |e_l^k(h)/h| \to 0, \quad \text{as } h \to 0.$$

The measure of the consistency order.

In real problems, we usuallly get the error estimates in the following form

$$\max_k |e_l^k(h)| \leq c \cdot h^{p+1}, \text{ for some } p \in [1, \infty).$$

In such a case we say that the local (truncated) error of the method is of the order $p + 1$.

2b. Another important property of the method is its **stability**. This notion corresponds to the continuous behavior of the solution, in other words if the initial data $\{y_0, y_1, \dots, y_{m-1}\}$ determining the whole approximate solution $\{y_k\}_{0 \leq k \leq n}$ are replaced with a bit perturbed data $\{\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{m-1}\}$, we obtain another a bit perturbed approximate solution $\{\bar{y}_k\}_{0 \leq k \leq n}$, $y_k \approx \bar{y}_k$. In such a case the approximate solution reflects main properties of the exact solution such as monotonicty or boundness.

Example 3.

$$y' = -my \ \ (m > 0), \ \ y(0) = 1,$$

The exact solution is $y(t) = e^{-mt}$.
Applying the explicit Euler method we have

$$y_{k+1} = y_k + hf(t_k, y_k), \ \ y_0 = 1$$
$$y_{k+1} = y_k + h(-my_k) = (1 - mh)y_k, \ \ y_0 = 1$$

Hence we obtain the approximate solution $y_k = y_0(1 - mh)^k$, $0 \le k \le n$. In this case the approximate solution reflects the properties of the exact solution if and only if

$$0 < 1 - mh < 1$$

which gives a bound for h

$$0 < h < 1/m,$$

The interval $h \in (0, 1/m)$ is called the interval of stability.

Example 4. We consider the same equation, but now we apply the implicit Euler method, which results in the following

$$y_{k+1} = y_k + h(-my_{k+1})$$
$$y_{k+1} = \frac{1}{1 + mh} y_k, \ \ y_0 = 1.$$

Hence, formula $y_k = y_0 \left(\frac{1}{1+mh}\right)^k$, $0 \le k \le n$ gives the approximate solution. We see that the approximate solution reflects the properties of the exact solution irrespective of the value of the parameter h.

Conclusion. We say that the stability domain of the explicit Euler method is $0 < h < 1/m$ and of the implicit one is $0 < h < \infty$.

Remark. Usually the implicit methods are more stable than the explicit ones.

Example 5a. Let us consider

$$\begin{cases} y'(t) = 5y(t) \\ y(0) = 1 \end{cases}$$

The solution of the problem is $y(t) = e^{5t}$. Applying the backwad implicit Euler method we get the approximate solution

$$y_0 = 1, \ y_n = \frac{1}{(1 - 5h)^n}.$$

Note that for $h > 2/5$ the approximate solution tends to zero, while the exact solution goes to ∞.

Example 5. A system of the differential equations

$$\begin{cases} y_1' = a\, y_1(t) + b\, y_2(t) \\ y_2' = c\, y_1(t) + d\, y_2(t) \\ y_1(0) = y_1^0, \ \ y_2(0) = y_2^0 \end{cases}$$

We write it in a concise vector form: $\vec{y}' = A\vec{y}(t)$, $\vec{y}(0) = (y_1^0, y_2^0)$, where

$$A = \begin{bmatrix} a, b \\ c, d \end{bmatrix}, \quad \vec{y}(t) = (y_1(t), y_2(t));$$

Further we simply write y instead of $\vec{y}$. The explicit Euler method gives

$$y_{k+1} = y_k + hAy_k = (I + hA)y_k$$

Hence, we obtain

$$y_k = (I + hA)^k y_0, \quad k = 1, 2, 3, \ldots, n.$$

To calculate $(I + hA)^k$ we use the Jordan form of the matrix A. For example, let

$$A = M^{-1}JM,$$

where J is the Jordan form of A and M is a similarity matrix. It is known that the matrix J has one of the following forms

$$(1) \qquad J = \begin{cases} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \\ \\ \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \\ \\ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad \lambda = \alpha \pm \beta i \end{cases}$$

where λ_1, λ_2 and λ are the eigenvalues of matrix A. Let's consider the first case with $\lambda_1, \lambda_2 < 0$. Then

$$(I + hA)^k = M^{-1}(I + hJ)^k M = M^{-1} \begin{bmatrix} (1 + h\lambda_1)^k & 0 \\ 0 & (1 + h\lambda_2)^k \end{bmatrix} M$$

As a conclusion we obtain

$$\begin{cases} \text{if } |1 + h\lambda_i| > 1, \text{ for some } i, & \text{then the method is unstable,} \\ \text{if } |1 + h\lambda_i| < 1, \text{ for } i = 1, 2, & \text{then the method is stable,} \end{cases}$$

Finally, the stability domain of the approximate method is $h \in (0, 2/\max(|\lambda_1|, |\lambda_2|))$.

Example 6. Stability in a case of a general method

$$y_{k+1} = a_{m-1}y_k + a_{m-2}y_{k-1} + \cdots + a_0 y_{k+1-m} + hF(t_k, h, y_{k+1}, y_k, \ldots, y_{k+1-m})$$

We first consider a test problem

$$\begin{cases} y' = 0 \\ y_0 = \alpha \end{cases}$$

In this case we have $f(t,y) \equiv 0$, which gives $F \equiv 0$ and the exact solution is $y(t) = \alpha$. Applying our algorithm we get a recurrence equation

$$(2) \quad y_{k+1} = \sum_{j=0}^{m-1} a_j y_{k+1-m+j}$$

to which we assign the characteristic polynomial

$$\rho(z) = z^m - a_{m-1}z^{m-1} - a_{m-2}z^{m-2} - \cdots - a_1 z - a_0.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of the equation $\rho(z) = 0$.

Since we have

$$y_0 = \alpha, \; y_1 = \alpha, \; \ldots, y_{m-1} = \alpha,$$

we also expect that $y_m = \alpha$. However, it follows from the algorithm that

$$y_m = \sum_{j=0}^{m-1} a_j y_j, \qquad \text{or} \qquad \alpha = \sum_{j=0}^{m-1} a_j \alpha$$

which gives

$$(3) \quad 1 = a_{m-1} + a_{m-2} + \cdots + a_1 + a_0$$

Conclusion

If the method is consistent with the equation, then (3) holds. In other words:

$\lambda_1 = 1$ is a solution of the equation $\rho(z) = 0$.

Now, it is known that the general solution of the recurrence equation (2) has the form

$$y_k = \sum_{i=1}^{m} c_i \lambda_i^k = c_1 \cdot 1 + \sum_{i=2}^{m} c_i \lambda_i^k$$

Since in our case $y_k = \alpha$, $k = 0, 1, 2, \ldots$, we conclude that

$$c_1 = \alpha, \quad c_2 = c_3 = \cdots = c_m = 0.$$

However, so far we did not take into account the process of the round-off errors in the computer calculations. In such a case we have to note that the real approximate solution is of the form

$$y_k = \alpha + \sum_{i=2}^{m} c_i \lambda_i^k.$$

Therefore we note that

1. if all roots of the characteristics equation $\rho(z) = 0$ satisfy the condition $|\lambda_i| < 1$, $i = 2, 3, \ldots, m$, then

$$\left| \sum_{i=2}^{m} c_i \lambda_i^k \right| \to 0,$$

as $k \to \infty$;

2. if $|\lambda_{i_0}| \geq 1$, for some $2 \leq i_0 \leq m$, then

$$\left|\sum_{i=2}^{m} c_i \lambda_i^k\right| \not\to 0,$$

Hence it follows that in the second case the scheme cannot produce the stable solution of the problem.
Although it seems that we have considered only an approximation in the very particular case, for the initial value problem

$$y' = 0, \quad y(0) = \alpha,$$

the stability characteristic remains the same also in a more general situation, when $f(t, y)$ is not identically zero. This is due the fact that the solution to the homogeneous equation is embeded in the solution to any equation.

Example 7. The initial problem as in example 3. To obtain an approximate solution, we apply the following scheme

$$y_{k+1} = -4y_k + 5y_{k-1} + h(4f(t_k, y_k) + 2f(t_{k-1}, y_{k-1}).$$

Now, $a_1 = -4$, $a_0 = 5$, so $\rho(z) = z^2 + 4z - 5$.
The characteristic equation $\rho(z) = 0$ has solutions $\lambda_1 = 1$, $\lambda_2 = 5$. Because $\lambda_2 > 1$ we conclude that this scheme is unstable.