Finite difference methods for parabolic problems.

Parabolic problems in bounded domains. The model problem

$$\begin{cases} u_t = \alpha^2 u_{xx} + f(x,t), & x \in (0,l),\ t > 0 \\ u(0,t) = u(l,t) = 0; & t > 0 \\ u(x,0) = v(x), & x \in (0,1) \end{cases}$$

Notation.

We introduce the grid in the strip $[0,l] \times R_+$ with width h in x variable and with the time step k. Denote the exact solution at mesh points by $u_i^n = u(x_i, t_n)$, $u^n = \{u_i^n\}$ and approximate solution by $U^n = \{U_i^n\}$, $U_i^n \approx u(x_i, t_n)$.

1. Forward Euler's scheme

$$\frac{U_i^{n+1} - U_i^n}{k} = \alpha^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + f_i^n, \quad i = 1, 2, \ldots, m-1$$

$$0 = x_0 < x_1 < \cdots < x_m = l, \quad 0 = t_0 < t_1 < \cdots < t_N = T$$

$$U_i^0 = V_i, \quad U_0^n = 0 = U_m^n, \quad i = 0, 1, \ldots, m$$

After some modification we obtain

$$U_i^{n+1} = \lambda U_{i+1}^n + (1 - 2\lambda) U_i^n + \lambda U_{i-1}^n + k f_i^n, \quad i = 1, 2, \ldots, m-1$$

where

$$\lambda = \alpha^2 \frac{k}{h^2},$$
$$U^n = [U_1^n, U_2^n, \ldots, U_{m-1}^n]'$$
$$f^n = [f_1^n, f_2^n, \ldots, f_{m-1}^n]', \text{ with } f_j^n = f(x_j, t_n),$$

or in the vector form

$$(1) \quad U^{n+1} = (I + \lambda A) U^n + k f^n$$

where

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and I is identity matrix of the size $m-1 \times m-1$. In the matrix form

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = A_\lambda \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{m-1}^n \end{bmatrix} + k \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{m-1}^n \end{bmatrix}$$

where

$$A_\lambda = I + \lambda A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 & \ldots & 0 \\ \lambda & 1-2\lambda & \lambda & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & 1-2\lambda \end{bmatrix}$$

This method is stable if $\lambda = \alpha^2 \frac{k}{h^2} \leq \frac{1}{2}$. The convergence result for this method

Theorem 3

$$||U^n - u^n||_\infty \leq C \cdot n \cdot k \cdot h^2 \max_{t \leq nk} ||u(x,t)||_{c^4}.$$

2. The backward Euler's method.

The stability requirement $k \leq \frac{1}{2}h^2$ used for the forward Euler's method is quite restrictive in practice and it would be desirable to relax it to be able to use h and k of the same order of magnitude. For this purpose one may define an implicit method by the backward Euler's scheme.

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + f_i^{n+1}, \quad i = 1, 2, \ldots, m-1$$
$$U_i^0 = V_i, \quad U_0^n = 0 = U_m^n, \quad i = 0, 1, \ldots, m$$

In the matrix form

$$(2) \quad (I - \lambda A)U^{n+1} = U^n + kf^{n+1}$$

or

$$B \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{m-1}^n \end{bmatrix} + k \begin{bmatrix} f_1^{n+1} \\ f_2^{n+1} \\ \vdots \\ f_{m-1}^{n+1} \end{bmatrix}$$

where

$$B = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 & \ldots & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda & 1+2\lambda \end{bmatrix}$$

B is diagonally dominant, symmetric and tridiagonal. Thus it is invertible and the vector $\left[ccccccU_1^{n+1},\ U_2^{n+1},\ \ldots,\ U_{m-1}^{n+1}\right]$ can be determined

The stability of the method does not require any relation between k and h.

The convergence results for the implicit method

Theorem 4

$$||U^n - u^n||_\infty \leq C \cdot n \cdot k(h^2 + k) \max_{t \leq T} |u(\cdot, t)|_{C^4}$$

Comments

The above convergence result for the backward Euler method is satisfactory in that it requires no restrictions on the mesh ratio $k/h^2 = \lambda$. On the other hand, since it is only first order accurate in time, the error in the time discretization will dominate unless k is chosen much smaller than h.

3. The Cranck-Nicolson scheme, as the stable second order accurate method. It can be obtained informally in the following way

$$\frac{U_j^{n+1}-U_j^n}{k} = \frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + f_j^n$$

$$\frac{U_j^{n+1}-U_j^n}{k} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} + f_j^{n+1}$$

Taking the average of the above equalities we obtain

$$\frac{U_j^{n+1}-U_j^n}{k} = \frac{1}{2}\left(\frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2}\right) + f_j^{n+\frac{1}{2}},$$

where $f_j^{n+\frac{1}{2}} = \frac{1}{2}(f_j^n + f_j^{n+1})$.

After some simple transformations

$$(3) \quad (1+\lambda)U_j^{n+1} - \frac{1}{2}\lambda\left(U_{j-1}^{n+1} + U_{j+1}^{n+1}\right) = (1-\lambda)U_j^n + \frac{1}{2}\lambda\left(U_{j-1}^n + U_{j+1}^n\right) + f_j^{n+\frac{1}{2}}$$

or

$$U^{n+1} - U^n = \frac{1}{2}\lambda A U^n + \frac{1}{2}\lambda A U^{n+1} + k f^{n+\frac{1}{2}}$$

Finally

$$\left(I - \frac{1}{2}\lambda A\right) U^{n+1} = \left(I + \frac{1}{2}\lambda A\right) U^n + k f^{n+\frac{1}{2}}$$

$$U_0^{n+1} = U_m^{n+1} = 0.$$

Introducing the matrices

$$B = I - \frac{1}{2}\lambda A = \begin{bmatrix} 1+\lambda & -\frac{1}{2}\lambda & 0 & 0 & \dots & 0 \\ -\frac{1}{2}\lambda & 1+\lambda & -\frac{1}{2}\lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{2}\lambda & 1+\lambda \end{bmatrix}$$

$$C = I + \frac{1}{2}\lambda A = \begin{bmatrix} 1-\lambda & \frac{1}{2}\lambda & 0 & 0 & \dots & 0 \\ \frac{1}{2}\lambda & 1-\lambda & \frac{1}{2}\lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2}\lambda & 1-\lambda \end{bmatrix}$$

the Cranck-Nicolson scheme can be written in the matrix form

$$B \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = C \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{m-1}^n \end{bmatrix} + k \begin{bmatrix} f_1^{n+\frac{1}{2}} \\ f_2^{n+\frac{1}{2}} \\ \vdots \\ f_{m-1}^{n+\frac{1}{2}} \end{bmatrix}$$

The stability condition in the $||\cdot||_\infty$ norm: $\lambda \leq 1$.

Remark

It can be shown that in the case of the norm $||\cdot||_2$ no restrictions on λ are needed.

The convergence result.

Theorem 5

For $\lambda \leq 1$

$||U^n - u^n||_\infty \leq C \cdot nk(h^2 + k^2) \max_{t \leq T} |u(\cdot, t)|_{C^4}$

For any $\lambda > 0$

$||U^n - u^n||_{2,h} \leq C \cdot nk(h^2 + k^2) \max_{t \leq T} |u(\cdot, t)|_{C^4}$

definition:

$$||w||_{2,h} = \left(h \sum_{j=0}^{m} w_j^2 \right)^{1/2}$$