4. The Fourier method for the diffusion problem

$$\begin{cases} u_t = \alpha^2 u_{xx} + f(x,t), \quad x \in (0,l),\ t>0 \\ u(0,t) = u(l,t) = 0; \quad t>0 \\ u(x,0) = v(x), \quad x \in (0,1) \end{cases}$$

1. We first observe that the functions $e^{-\frac{\alpha^2\pi^2}{l^2}t}\sin\left(\frac{\pi x}{l}\right)$, $e^{-4\frac{\alpha^2\pi^2}{l^2}t}\sin\left(2\frac{\pi x}{l}\right)$, $e^{-9\frac{\alpha^2\pi^2}{l^2}t}\sin\left(3\frac{\pi x}{l}\right)$, $\ldots$, $e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin\left(k\frac{\pi x}{l}\right)$, $\ldots k = 1,2,3,\ldots$ satisfy the homogeneous problem

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \end{cases}$$

and the boundary condition

$$u(0,t) = u(l,t) = 0$$

Moreover, each sum

$$\sum_{k=1}^{\infty} a_k e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin\left(k\frac{\pi x}{l}\right),$$

is also a solution to that problem if the sequence $\{a_k\}$ is sufficiently fast convergent to zero.

2. We are looking for solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin(k\pi\frac{x}{l})$$

If this series is sufficiently fast convergent, then the function $u(x,t)$ can be considered as the solution of the problem. In order to determine the coefficients $a_k(t)$ we make the following observations

(1) $$u_t(x,t) = \sum_{k=1}^{\infty} a_k'(t) e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right) + \sum_{k=1}^{\infty} -k^2\frac{\alpha^2\pi^2}{l^2} a_k(t) e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right)$$

(2) $$u_{xx}(x,t) = \sum_{k=1}^{\infty} -\frac{k^2\pi^2}{l^2} a_k(t) e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right)$$

(3) $$f(x,t) = \sum_{k=1}^{\infty} f_k(t)\sin\left(k\pi\frac{x}{l}\right)$$

where the last expansion is obtained from the general formulas for the Fourier expansion series

(4) $$f_k(t) = \int_0^l f(x,t)\sin\left(k\pi\frac{x}{l}\right)dx \Big/ \int_0^l \sin^2\left(k\pi\frac{x}{l}\right)dx = \frac{2}{l}\int_0^l f(x,t)\sin\left(k\pi\frac{x}{l}\right)dx$$

Now from the differential equation it follows that

$$\sum_{k=1}^{\infty} a_k'(t)e^{-\frac{k^2\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right)+\sum_{k=1}^{\infty}-\frac{k^2\alpha^2\pi^2}{l^2}a_k(t)e^{-\frac{k^2\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right)=$$
$$\alpha^2\sum_{k=1}^{\infty}-\frac{k^2\pi^2}{l^2}a_k(t)e^{-\frac{k^2\alpha^2\pi^2}{l^2}t}\sin\left(k\pi\frac{x}{l}\right)+\sum_{k=1}^{\infty}f_k(t)\sin\left(k\pi\frac{x}{l}\right)$$

A comparison of the terms in the sums above we obtain

(5) $$a_k'(t)=f_k(t)e^{\frac{k^2\alpha^2\pi^2}{l^2}t},\quad k=1,2,\ldots$$

Finally

(6) $$a_k(t)=\int_0^t f_k(s)e^{\frac{k^2\alpha^2\pi^2}{l^2}s}ds+a_k(0),\quad k=1,2,\ldots$$

The coefficients $a_k(0)$ are retrieved from the initial condition as follows

$$v(x)=\sum_{k=1}^{\infty}v_k\sin\left(k\pi\frac{x}{l}\right),$$

(7) $$v_k=\int_0^l v(x)\sin\left(k\pi\frac{x}{l}\right)dx\Big/\int_0^l\sin^2\left(k\pi\frac{x}{l}\right)dx=\frac{2}{l}\int_0^l v(x)\sin\left(k\pi\frac{x}{l}\right)dx,$$

$$a_k(0)=v_k,\quad k=1,2,3,\ldots$$

Remark

In practise, the integral in (7) is calculated approximately by using for example the complex Simpson method

(8) $$\int_0^l f(s)ds\approx\frac{1}{6}h\sum_{i=0}^{n}\left(f(x_i)+4f(x_{i+\frac{1}{2}})+f(x_{i+1})\right)$$

where $0=x_0\leq x_1\leq x_2\leq\ldots x_{n+1}=l$, $h=(x_{i+1}-x_i)/n$, $x_{i+\frac{1}{2}}=\frac{x_i+x_{i+1}}{2}$.

Example. The Fourier method for the following diffusion problem.

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad x \in (0, l),\ t > 0 \\ u_x(0,t) = u(l,t) = 0; \quad t > 0 \\ u(x,0) = v(x), \quad x \in (0,1) \end{cases}$$

1. First we consider the eigenvalue problem associated with the differential equation

$$(9) \qquad \begin{cases} v''(x) = \lambda v(x) \\ v'(0) = v(l) = 0 \end{cases}$$

In this problem we are looking for unknown nontrivial functions $v(x)$ (the eigenfunctions) and numbers λ (the eigenvalue).

The characteristic equation associated to this differential problem has the following form

$$\alpha^2 z^2 = \lambda$$

Respectively to possible expected values of λ we have to consider the following cases:

a) $\lambda > 0$: then the characteristic equation has two real solutions: $z = \pm\frac{1}{\alpha}\sqrt{\lambda}$. Consequently the general solution to the differential equation has the form

$$v(x) = Ae^{-\frac{1}{\alpha}\sqrt{\lambda}x} + Be^{-\frac{1}{\alpha}\sqrt{\lambda}x}$$

The function $v(x)$ has to satisfy the homogeneous boundary condition $v'(0) = v(l) = 0$, wich results in equations

$$\begin{cases} -\frac{1}{\alpha}\sqrt{\lambda}A + B = 0 \\ Ae^{-\frac{1}{\alpha}\sqrt{\lambda}l} + Be^{-\frac{1}{\alpha}\sqrt{\lambda}l} = 0 \end{cases}$$

This system of linear equations has only one trivial solution $A = B = 0$.

b) $\lambda = 0$: then the characteristic equation has only the trivial solution: $z = 0$. Consequently the general solution to the differential equation has the form

$$v(x) = Ax + B$$

The function $v(x)$ has to satisfy the homogeneous boundary condition $v'(0) = v(l) = 0$, wich results in equations

$$\begin{cases} A = 0 \\ A \cdot l + B = 0 \end{cases}$$

This system of linear equations has only one solution $A = B = 0$.

c) $\lambda < 0$: then the characteristic equation has two imaginary solutions: $z = \pm\frac{1}{\alpha}i\sqrt{-\lambda}$. Consequently the general solution to the differential equation has the form

$$v(x) = A\cos\left(\frac{1}{\alpha}\sqrt{-\lambda}x\right) + B\sin\left(\frac{1}{\alpha}\sqrt{-\lambda}x\right)$$

The function $v(x)$ has to satisfy the homogeneous boundary condition $v'(0) = v(l) = 0$, which results in equations

$$\begin{cases} \frac{1}{\alpha}\sqrt{-\lambda}B \cdot 1 = 0, \text{ which gives } B = 0 \\ A \cdot \cos(\frac{1}{\alpha}\sqrt{-\lambda}l) = 0, \\ \text{which gives } \frac{1}{\alpha}\sqrt{-\lambda}l = (k+\frac{1}{2})\pi, \ k = 0, 2, 3, \ldots \end{cases}$$

Hence

the eigenvalues: $\lambda_k = -(k+\frac{1}{2})^2\alpha^2\pi^2/l^2$, $k = 0, 2, 3, \ldots$

the eigenfunctions: $v_k(x) = \cos\left((k+\frac{1}{2})\frac{\pi x}{l}\right)$, $k = 0, 2, 3, \ldots$

Now if $u(x,t) = e^{\lambda_k t}v_k(x)$ then

$$u_t = \lambda_k e^{\lambda_k t} v_k(x) = \lambda_k u$$

$$u_{xx} = e^{\lambda_k t} v_k''(x) = \frac{1}{\alpha^2}\lambda_k e^{\lambda_k t} v_k(x) = \frac{1}{\alpha^2}\lambda_k u(x) \tag{10}$$

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u_x(0) = u(l) = 0 \end{cases} \tag{11}$$

In this way we get the series of solutions

$$e^{\lambda_k t}\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right), \quad \lambda_k = -\left(k+\frac{1}{2}\right)^2\frac{\alpha^2}{l^2}$$

2.

$e^{-\frac{\alpha^2\pi^2}{l^2}t}\cos\left(\frac{\pi x}{l}\right)$, $e^{-4\frac{\alpha^2\pi^2}{l^2}t}\cos\left(2\frac{\pi x}{l}\right)$, $e^{-9\frac{\alpha^2\pi^2}{l^2}t}\cos\left(3\frac{\pi x}{l}\right)$, $\ldots$, $e^{-k^2\frac{\alpha^2\pi^2}{l^2}t}\cos\left(k\frac{\pi x}{l}\right)$,
$\ldots k = 1, 2, 3, \ldots$

We are looking for a solution of the following form

$$u(x,t) = \sum_{k=0}^{\infty} a_k e^{\lambda_k t}\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right),$$

To find unknown coefficients, we first expand $u(x,0)$ into the following Fourier series

$$u(x,0) = v(x) = \sum_{k=0}^{\infty} v_k \cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right)$$

$$v_k = \frac{2}{l}\int_0^l v(x)\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right)dx, \ k = 0, 1, 2, 3, \ldots$$

On the other hand

$$u(x,0) = \sum_{k=0}^{\infty} a_k \cos\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{l}\right)$$

Hence it follows that $a_k = v_k$, $k = 0, 1, 2, 3, \ldots$ which gives

$$u(x,t) = \sum_{k=0}^{\infty} v_k e^{\lambda_k t} \cos\left(\left(k + \frac{1}{2}\right)\frac{\pi x}{l}\right).$$