The Fourier method for the hyperbolic differential equations.

Let us consider the wave equation as the model problem.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad t > 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = h(x), \quad 0 \le x \le l, \\ \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l, \end{cases}$$

where $f(x)$, $g(x)$, $h(x)$ are given smooth bounded functions.

1. We are looking for a solution of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin\left(k\frac{\pi x}{l}\right).$$

The the function series $\left\{\sin\left(k\frac{\pi}{l}x\right)\right\}$ has been chosen because each function $\sin\left(k\frac{\pi}{l}\right)$ is a nontrivial solution of the auxilliary problem

$$\begin{cases} w''(x) = \lambda w(x) \\ w(0) = w(l) = 0. \end{cases}$$

for $\lambda = \lambda_k = -k^2\frac{\pi^2}{l^2}$, $k = 1, 2, 3, \ldots$. Moreover this is an orthogonal function series, which means that

$$\int_0^{\pi} \sin\left(m\frac{\pi}{l}\right) \sin\left(n\frac{\pi}{l}x\right) \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0 (= \frac{l}{2}), & \text{if } m = n \end{cases}$$

If the sequence of coefficients $\{a_k(t)$ is sufficiently fast convergent to zero, then the function $u(x,t)$ can be considered as the solution of the problem. In order to determine the coefficients $a_k(t)$ we make the observations

$$(1) \quad u_{tt}(x,t) = \sum_{k=1}^{\infty} a_k''(t) \sin\left(k\frac{\pi x}{l}\right)$$

$$(2) \quad u_{xx}(x,t) = \sum_{k=1}^{\infty} -\frac{k^2\pi^2}{l^2} a_k(t) \sin\left(k\frac{\pi x}{l}\right)$$

$$(3) \quad f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin\left(k\frac{\pi x}{l}\right)$$

where coefficients $f_k(t)$ are obtained from the general formulas for the Fourier expansion series

$$(4) \quad f_k(t) = \int_0^l f(x,t) \sin(k\pi\frac{x}{l})dx \Big/ \int_0^l \sin^2(k\pi\frac{x}{l})dx = \frac{2}{l}\int_0^l f(x,t)\sin(k\pi\frac{x}{l})dx$$

Now from the differential equation it follows that

$$\sum_{k=1}^{\infty} a_k''(t) \sin(k\pi\frac{x}{l}) =$$

$$\alpha^2 \sum_{k=1}^{\infty} -\frac{k^2\pi^2}{l^2} a_k(t) \sin(k\pi\frac{x}{l}) + \sum_{k=1}^{\infty} f_k(t) \sin(k\pi\frac{x}{l})$$

Comparing the terms in the sum above we obtain differential equations

$$(5)\quad a_k''(t) = -\frac{k^2\alpha^2\pi^2}{l^2}a_k(t) + f_k(t), \quad k = 1, 2, \ldots$$

which have the following solutions

$$(6)\quad a_k(t) = a_k(0)\cos\left(\frac{k\alpha\pi}{l}t\right) + \frac{l}{k\alpha\pi}a_k'(0)\sin\left(\frac{k\alpha\pi}{l}t\right) +$$

$$\frac{l}{k\alpha\pi}\int_0^t f_k(s)\sin\left(\frac{k\alpha\pi}{l}(t-s)\right)ds, \quad k = 1, 2, \ldots$$

The coefficients $a_k(0)$ are retrieved from the initial condition as follows

$$h(x) = \sum_{k=1}^{\infty} h_k \sin\left(k\pi\frac{x}{l}\right),$$

where

$$(7)\quad h_k = \int_0^l h(x)\sin\left(k\pi\frac{x}{l}\right)dx \Big/ \int_0^l \sin^2\left(k\pi\frac{x}{l}\right)dx = \frac{2}{l}\int_0^l h(x)\sin\left(k\pi\frac{x}{l}\right)dx,$$

which gives

$$a_k(0) = h_k, \quad k = 1, 2, 3, \ldots$$

the coefficients $a_k'(0)$ are retrieved from the initial condition as follows

$$g(x) = \sum_{k=1}^{\infty} g_k \sin\left(k\pi\frac{x}{l}\right),$$

where

$$(8)\quad g_k = \int_0^l g(x)\sin\left(k\pi\frac{x}{l}\right)dx \Big/ \int_0^l \sin^2\left(k\pi\frac{x}{l}\right)dx = \frac{2}{l}\int_0^l g(x)\sin\left(k\pi\frac{x}{l}\right)dx,$$

which gives

$$a_k'(0) = g_k, \quad k = 1, 2, 3, \ldots$$

Remark

In practise, the integrals in (4), (6), (7) and (8) are calculated approximately by using for example the complex Simpson method

$$(9)\quad \int_0^l f(s)ds \approx \frac{1}{6}h\sum_{i=0}^{n}\left(f(x_i) + 4f(x_{i+\frac{1}{2}}) + f(x_{i+1})\right)$$

where $0 = x_0 \le x_1 \le x_2 \le \ldots x_{n+1} = l$, $h = (x_{i+1} - x_i)/n$, $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$.