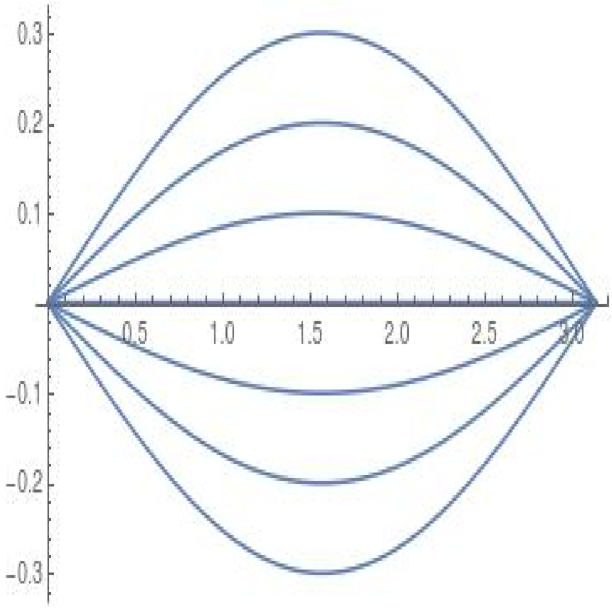
The finite difference methods for the hyperbolic differential equations.

Let us consider the wave equation as the model problem.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad t > 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = h(x), \quad 0 \le x \le l, \\ \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l, \end{cases}$$

where $f(x)$, $g(x)$, $h(x)$ are given smooth bounded functions.

Such equations describe the wave phenomena like the movement of the string or the propagation of the sound wave in gases.



1. The forward explicit Euler's scheme produces the sequence of approximate values for the solution by relations

$$\frac{U_i^{n-1} - 2U_i^n + U_i^{n+1}}{k^2} = \alpha^2 \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2} + f_i^n,$$

where $f_i^n = f(x_i, t_n)$. If $\lambda = \alpha \frac{k}{h}$, we can write this difference equation in the vector form

(1) $\quad IU^{n-1} - 2IU^n + IU^{n+1} = \lambda^2 AU^n + k^2 f^n$

where $U^n = [U_1^n, U_2^n, \dots, U_{m-1}^n]'$, similarly U^{n-1} and U^{n+1}, $f^n = [f_1^n, f_2^n, \dots, f_{m-1}^n]'$,

(2) $$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and I is identity matrix of the size $(m-1) \times (m-1)$. After a modification

(3) $\quad U^{n+1} = (2I + \lambda^2 A)U^n - IU^{n-1} + k^2 f^n.$

This equation holds for each $n = 1, 2, \ldots$. The boundary conditions give

(4) $\quad U_0^n = U_m^n = 0,$

for each $n = 1, 2, 3, \ldots$, and the initial condition implies that

(5) $\quad U_i^0 = h(x_i)$

for each $i = 1, 2, \ldots, m-1$.
Writing (3) in a matrix form we obtain

$$(6)\quad \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{m-1}^n \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2 \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{m-1}^n \end{bmatrix}$$

Equations (3) (or (6)) imply that the $(n+1)$-st time step requires values from n-th and $(n-1)$-st time steps. This produces a minor starting problem since the values for $n = 0$ are given by equation (5), but the values for $n = 1$, which are needed in equation (3) to compute U_i^2 must be obtained from the initial velocity condition

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 \leq x \leq l.$$

The first approach is to replace $(\partial u/\partial t)$ by a forward-difference approximation, which results in

(7) $\quad U_i^1 = U_i^0 + kg(x_i)$

A better approximation to $u(x_i, t_1)$ can be rather easily obtained particularly when the second derivative of f at x_i can be determined.

(8) $\quad U_i^1 = U_i^0 + kg(x_i) + \dfrac{\alpha^2 k^2}{2} h''(x_i).$

This is an approximation with the local truncation error $O(k^2)$ for each $i = 1, 2, \ldots, m-1$. If $h \in C^4[0, 1]$ but $h''(x_i)$ is not readily available we can use an approximation

(9) $\quad U_i^1 = (1 - \lambda^2) h(x_i) + \dfrac{\lambda^2}{2} h(x_{i+1}) + \dfrac{\lambda^2}{2} h(x_{i-1}) + kg(x_i),$

to approximate U_i^1 for each $i = 1, 2, \ldots, m-1$.

2. The backward implicit Euler's scheme produces the sequence of approximate values for the solution by relations

$$\frac{U_i^{n-1} - 2U_i^n + U_i^{n+1}}{k^2} = \alpha^2 \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{h^2} + f_i^{n+1}$$

If $\lambda = \alpha\frac{k}{h}$, we can write the difference equation as

$$U^{n-1} - 2U^n + U^{n+1} = \lambda^2 AU^{n+1} + k^2 f^{n+1}$$

or

$$(I - \lambda^2 A)U^{n+1} = 2U^n - U^{n-1} + k^2 f^{n+1}. \tag{10}$$

This equation holds for each $n = 1, 2, \ldots$. The boundary conditions give

$$U_0^n = U_m^n = 0,$$

for each $n = 1, 2, 3, \ldots$, and the initial condition implies that

$$U_i^0 = h(x_i)$$

for each $i = 1, 2, \ldots, m-1$. Vector $[U_1^1, U_2^1, \ldots, U_{m-1}^1]$ is calculated using one of the formula (7), (8) or (9).
Writing relation (10) in a matrix form we obtain

$$B \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = 2 \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{m-1}^{n} \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2 \begin{bmatrix} f_1^{n+1} \\ f_2^{n+2} \\ \vdots \\ f_{m-1}^{n+1} \end{bmatrix} \tag{11}$$

where

$$B = \begin{bmatrix} 1+2\lambda^2 & -\lambda^2 & 0 & 0 & \ldots & 0 \\ -\lambda^2 & 1+2\lambda^2 & -\lambda^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -\lambda^2 & 1+2\lambda^2 \end{bmatrix}$$

3. The Cranck-Nicolson scheme, as the stable second order accurate method. It can be obtained informally in the following way

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + f_j^n$$
$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} + f_j^{n+1}$$

Taking the average of the above equalities we obtain

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \alpha^2 \frac{1}{2} \left(\frac{U_{j-1}^n - 2U_j^n + U_j^n}{h^2} + \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} \right) + f_j^{n+\frac{1}{2}},$$

where $f_j^{n+\frac{1}{2}} = \frac{1}{2}(f_j^n + f_j^{n+1})$.
After some simple transformations we obtain

$$U^{n+1} - 2U^n + U^{n-1} = \frac{1}{2}\lambda^2 AU^n + \frac{1}{2}\lambda^2 AU^{n+1} + k^2 f^{n+\frac{1}{2}}$$

or finally

$$(12) \quad \left(I - \frac{1}{2}\lambda^2 A\right) U^{n+1} = \left(2I + \frac{1}{2}\lambda^2 A\right) U^n - U^{n-1} + k^2 f^{n+\frac{1}{2}}$$

$$U_0^{n+1} = U_m^{n+1} = 0,$$

where A is the earlier introduced matrix in (2). Introducing the matrices

$$B = I - \frac{1}{2}\lambda^2 A = \begin{bmatrix} 1+\lambda^2 & -\frac{1}{2}\lambda^2 & 0 & 0 & \dots & 0 \\ -\frac{1}{2}\lambda^2 & 1+\lambda^2 & -\frac{1}{2}\lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{2}\lambda^2 & 1+\lambda^2 \end{bmatrix}$$

$$C = 2I + \frac{1}{2}\lambda^2 A = \begin{bmatrix} 2-\lambda^2 & \frac{1}{2}\lambda^2 & 0 & 0 & \dots & 0 \\ \frac{1}{2}\lambda^2 & 2-\lambda^2 & \frac{1}{2}\lambda^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2}\lambda^2 & 2-\lambda^2 \end{bmatrix}$$

the Cranck-Nicolson scheme can be written in the matrix form

$$B \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \end{bmatrix} = C \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{m-1}^{n} \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \\ \vdots \\ U_{m-1}^{n-1} \end{bmatrix} + k^2 \begin{bmatrix} f_1^{n+\frac{1}{2}} \\ f_2^{n+\frac{1}{2}} \\ \vdots \\ f_{m-1}^{n+\frac{1}{2}} \end{bmatrix}$$