Lecture 07. Statistics and probability theory.

Example. Formulas for the probability of the sum of the events.

- $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$,

  in the case of disjoint events, $A \cap B = \emptyset$ this formula takes the simplified form $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

- $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - (\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C)) + \mathrm{P}(A \cap B \cap C)$

- $\mathrm{P}(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_n) -$
  $(\mathrm{P}(A_1 \cap A_2) + \mathrm{P}(A_1 \cap A_3) + \cdots + \mathrm{P}(A_i \cap A_j) + \dots) +$
  $(\mathrm{P}(A_1 \cap A_2 \cap A_3) + \mathrm{P}(A_1 \cap A_3 \cap A_4) + \cdots + \mathrm{P}(A_i \cap A_j \cap A_k) + \dots) -$
  $\dots$

Exercise 1. We put randomly n letters into n empty addressed envelopes (one letter into one envelope). What is the probability that no letter matched the envelope?

Odp. 1. For simplicity we assume that the letters are labeled with number from 1 to n. Let A_i denote the event, that the letter i is in a proper envelope. The event $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ denotes that at least one letter matches an envelope. The event $A = (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)^c$ denotes that no letter matched the envelope. We pass to calculations:

$$P(A) = 1 - P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)$$

$$P(A_i) = \frac{(n-1)!}{n!}$$

$$P(A_1) + P(A_2) + \dots + P(A_n) = n\frac{(n-1)!}{n!} = \binom{n}{1}\frac{(n-1)!}{n!}\frac{1}{1!}$$

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!}$$

$$P(A_1 \cap A_2) + P(A_1 \cap A_3) + \dots + P(A_i \cap A_j) +$$
$$\dots + P(A_{n-1} \cap A_n) = \binom{n}{2}\frac{(n-2)!}{n!} = \frac{1}{2!}$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}$$

$$P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + \dots + P(A_i \cap A_j \cap A_k) +$$
$$\dots + P(A_{n-2} \cap A_{n-1} \cap A_n) = \binom{n}{3}\frac{(n-3)!}{n!} = \frac{1}{3!}$$

$$\vdots$$

$$P(A) = 1 - \left(\binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} + \dots\right) =$$
$$1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots\right)$$

Exercise 2. There are n balls labeled with numbers from 1 to n. We put them randomly into m drawers labeled with numbers from 1 to m. What is the probability that no drawer is empty, we assume that $m \le n$.

Answer. Let A_i be an event that the drawer with a label i is empty. The event $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m$ denotes that at least one drawer is empty. The event $A = (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m)^c$ denotes that no drawer is empty. We pass to calculations

$$P(A) = 1 - P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m)$$

$$P(A_i) = \frac{(m-1)^n}{m^n} = \left(1 - \frac{1}{m}\right)^n,$$

$$P(A_1) + P(A_2) + \cdots + P(A_m) = \binom{m}{1}\left(1 - \frac{1}{m}\right)^n$$

$$P(A_i \cap A_j) = \frac{(m-2)^n}{m^n} = \left(1 - \frac{2}{m}\right)^n,$$

$$P(A_1 \cap A_2) + P(A_1 \cap A_3) + \cdots + P(A_i \cap A_j) +$$

$$\cdots + P(A_{m-1} \cap A_m) = \binom{m}{2}\left(1 - \frac{2}{m}\right)^n$$

$$P(A_i \cap A_j \cap A_k) = \frac{(m-3)^n}{m^n} = \left(1 - \frac{3}{m}\right)^n$$

$$P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + \cdots + P(A_i \cap A_j \cap A_k) +$$

$$\cdots + P(A_{m-2} \cap A_{m-1} \cap A_m) = \binom{m}{3}\left(1 - \frac{3}{m}\right)^n$$

$$\vdots$$

$$P(A) = 1 - \left(\binom{m}{1}\left(1 - \frac{1}{m}\right)^n - \binom{m}{2}\left(1 - \frac{2}{m}\right)^n + \binom{m}{3}\left(1 - \frac{3}{m}\right)^n + \right.$$

$$\left. \cdots + (-1)^{m+1}\binom{m}{m}\right)$$

The conditional and total probability.

Example. In a batch of 1000 bulbs, the possible number of defective bulbs is $0, 1, 2, 3, 4$, or 5, all these numbers of defective bulbs are equally likely. We randomly select 100 light bulbs. What is the probability that none of the bulbs drawn are defective?

Answer. Let H_k, $k = 0, 1, 2, \ldots, 5$ be hipoteses that in the batch there are k defective bulbs, respectively. Hipoteses H_k exhaust all possible cases and $H_i \cap H_j = \emptyset$, $i \neq j$. Let A be an event that none of the bulbs drawn are defective. Then

$$\mathrm{P}(H_k) = \frac{1}{6}, \quad \mathrm{P}(A|H_k) = \binom{1000-k}{100} \frac{1}{\binom{1000}{100}}, \quad k = 0, 1, 2, \ldots, 5$$

The total probability

$$\mathrm{P}(A) = \mathrm{P}(A|H_0) \cdot \mathrm{P}(H_0) + \mathrm{P}(A|H_1) \cdot \mathrm{P}(H_1) + \cdots + \mathrm{P}(A|H_5) \cdot \mathrm{P}(H_5) =$$
$$\frac{1}{\binom{1000}{100}} \sum_{k=0}^{5} \binom{1000-k}{100} \cdot \frac{1}{6}.$$

Example. The urn contains black and white balls, in total n balls. It has been added a white ball to the urn. What is the probability of a white ball being drawn from the urn, if all hipotheses on the initial proportion of balls in the urn are equally likely.

Answer. Let H_k, $k = 0, 1, 2, \ldots, n$ be hipoteses that at the beginning the urn contained k - white balls respectively. Hipoteses H_k exhuast all possible cases and $H_i \cap H_j = \emptyset$. Let A be an event that the white ball was drawn. Then

$$\mathrm{P}(H_k) = \frac{1}{n+1}, \quad \mathrm{P}(A|H_k) = \frac{k+1}{n+1}, \quad k = 0, 1, 2, \ldots, n$$

Total probability is

$$\mathrm{P}(A) = \mathrm{P}(A|H_0) \cdot \mathrm{P}(H_0) + \mathrm{P}(A|H_1) \cdot \mathrm{P}(H_1) + \cdots + \mathrm{P}(A|H_n) \cdot \mathrm{P}(H_n) =$$
$$\left(\frac{1}{n+1} + \frac{2}{n+1} + \cdots + \frac{n+1}{n+1}\right) \frac{1}{n+1} = \frac{(n+1)(n+2)}{2} \frac{1}{(n+1)^2} = \frac{n+2}{2(n+1)}.$$

Random variable

Let S be a sample space and let P be a probability in S. Any numerical function

$$X : \ S \to \mathbb{R}$$

is called the random variable. We usually denote random variables by capital letters near the end of the alphabet, such as X or Y.

1. Discrete random variable

A **discrete random variable** X has a finite number of possible values.

The **probability distribution** of X lists the values and their probabilities:

Value of X	x_1	x_2	x_3	$\dots$	x_n
Probability	p_1	p_2	p_3	$\dots$	p_n

where the probabilities $p_i = \mathrm{P}(\{\omega : X(\omega) = x_i\})$, $i = 1, 2, 3, \dots, n$.

The probabilities p_i must satisfy two requirements:

1. Every probability p_i is a number between 0 and 1.

2. $p_1 + p_2 + \;\; + p_n = 1$.

Another look at distribution of the random variable X: we are interested in $\mathrm{P}(X \in A)$ for $A \subseteq R$,

$$\mathrm{P}(X \in A) = \sum_{x_k \in A} p_k,$$

where $A \subseteq R$, in particular

$$\mathrm{P}(X \in (a, b)) = \mathrm{P}(a < X < b) = \sum_{a < x_k < b} p_k$$

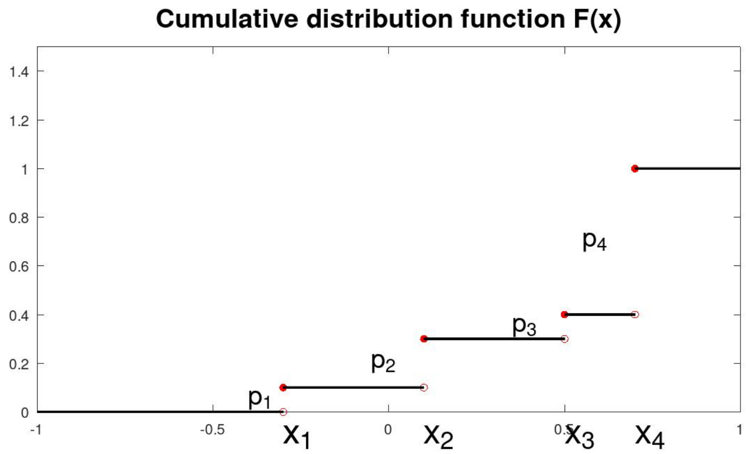
When A is the infinite interval $(-\infty, x]$ for any real number x, we can introduce a function of x as follows:

$$F(x) = \mathrm{P}(X \leq x) = \sum_{x_k \leq x} p_k$$

This function $x \to F(x)$ is called a **cumultative distribution function.**

Formula for the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < x_1 \\ p_1 & \text{for } x_1 \leq x < x_2 \\ p_1 + p_2 & \text{for } x_2 \leq x < x_3 \\ p_1 + p_2 + p_3 & \text{for } x_3 \leq x < x_4 \\ \vdots & \vdots \\ p_1 + p_2 + \dots + p_{n-1} & \text{for } x_{n-1} \leq x < x_n \\ 1 & \text{for } x_n \leq x < \infty \end{cases}$$



Let you note that $F(x)$ has the following properties:

- $0 \le F(x) \le 1$ for $x \in R$
- if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$, $F(x)$ is nodecreasiing
- $\lim_{x \to x_0+} F(x) = F(x_0)$, $F(x)$ is right-side continuous

Corollary

- $\mathrm{P}(X \in (a, b]) = \mathrm{P}(a < X \le b) = F(b) - F(a)$
- $\mathrm{P}(X = b) = \mathrm{P}(X \in \{b\}) = \lim_{x \to b+} F(x) - \lim_{x \to b-} F(x) = F(b+) - F(b-) = F(b) - F(b-)$
- since $[a, b]$=$\{a\} \cup (a, b]$,

$$\mathrm{P}(X \in [a, b]) = \mathrm{P}(X = a) + \mathrm{P}(X \in (a, b]) = (F(a) - F(a-)) + (F(b) - F(a)) = F(b) - F(a-)$$

- since $(a.b) \cup \{b\} = (a, b]$, $\mathrm{P}(X \in (a, b)) + \mathrm{P}(X = b) = \mathrm{P}(X \in (a, b])$ or

$$\mathrm{P}(X \in (a, b)) = (F(b) - F(a)) - (F(b) - F(b-)) = F(b-) - F(a)$$

Example. Grade distributions. North Carolina State University posts the grade distributions for its courses online. Students in one section of English in the spring 2006 semester received 31% A's, 40% B's, 20% C's, 4% D's, and 5% F's. Choose an English student at random. To "choose at random" means to give every student the same chance to be chosen. The student's grade on a four-point scale (with $A = 4$ to $F = 0$) is a random variable X.

The value of X changes when we repeatedly choose students at random, but it is always one of $0, 1, 2, 3$, or 4. Here is the distribution of X:

Value of X	0	1	2	3	4
Probability	0.05	0.04	0.20	0.40	0.31

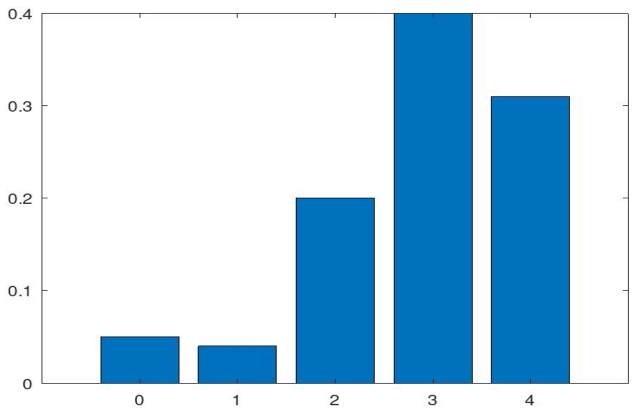
The probability that the student got a B or better is the sum of the probabilities of an A and a B. In the language of random variables,

$$P(X \geq 3) = P(X = 3) + P(X = 4) = 0.40 + 0.31 = 0.71.$$

$X \geq 3$ is a shorthand for $\{\omega : \ X(\omega) \geq 3\}$, similarly $X = 3$, $X = 4$

$\{\omega : \ X(\omega) \geq 3\} = \{\omega : \ X(\omega) = 3\} \cup \{\omega : \ X(\omega) = 4\}$

We can use histograms to show probability distributions as well as distributions of data



Example. Number of heads in four tosses of a coin. What is the probability distribution of the discrete random variable X that counts the number of heads in four tosses of a coin? We can derive this distribution if we make two reasonable assumptions:

• The coin is balanced, so it is fair and each toss is equally likely to give H or T.

• The coin has no memory, so tosses are independent.

The outcome of four tosses is a sequence of heads and tails such as HTTH.

There are 16 possible outcomes in all. These outcomes are equally likely. Each of the 16 possible outcomes has probability $1/16$. The number of heads X has possible values$0, 1, 2, 3,$ and 4. These values are not equally likely.

$X = 0$ for (T, T, T, T), $\mathrm{P}(X = 0) = \frac{1}{16}$

$X = 1$ for (H, T, T, T), (T, H, T, T), (T, T, H, T) and (T, T, T, H), $\mathrm{P}(X = 1) = \frac{4}{16}$

$X = 2$ for (H, H, T, T), (H, T, H, T), (H, T, T, H), (T, H, H, T), (T, H, T, H) and (T, T, H, H), $\mathrm{P}(X = 2) = \frac{6}{16}$

$X = 3$ for (H, H, H, T), (H, H, T, H), (H, T, H, H) and (T, H, H, H), $\mathrm{P}(X = 4) = \frac{4}{16}$

$X = 4$ for (H, H, H, H), $\mathrm{P}(X = 4) = \frac{1}{16}$

Value of X	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The fundamental characteristics of the random variable X:

- **Mean value (Expected value):** $\mu = \mathrm{E}X = \sum\limits_{\omega \in S} X(\omega)\mathrm{P}(\omega) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$;
- **Variance:** $\mathrm{D}^2[X] = \sigma^2 = \mathrm{E}(X - \mu)^2 = (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + (x_3 - \mu)^2 p_3 + \cdots + (x_n - \mu)^2 p_n$, σ is called the standard deviation of X.

Comments

Let us explicitly state a general method of calculating $\mathrm{E}X = \sum\limits_{\omega \in S} X(\omega)$ which is often expedient. Suppose the sample space S can be decomposed into disjoint sets A_k:

$$\bigcup_k A_k$$

in such a way that X takes the same value on each A_k. Thus we may write

$$X(\omega) = x_k, \text{ for } \omega \in A_k$$

where the x_k's need not all be different. We then have

$$\mathrm{E}X = \sum_k P(A_k) \cdot x_k = \sum_k P(X = x_k) \cdot x_k = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum_k x_k p_k$$

in the case of equally likely $x_k' s$, $k = 1, 2, 3, \dots, n$ all probababilities are equal $p_1 = p_2 = p_3 = \cdots = p_n = \frac{1}{n}$. Then

$$\mu = \mathrm{E}X = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots + x_n p_n =$$

$$\frac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n) = \bar{x}$$

$$\sigma^2 = (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + (x_3 - \mu)^2 p_3 + \cdots + (x_n - \mu)^2 p_n =$$

$$\frac{1}{n}\left((x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2\right) =$$

$$\frac{n-1}{n} \cdot s^2.$$

the mean value $\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$ is equal to $\mathrm{E}X$.

Arithmetic operations on the random variables. Let X and Y be to random variables on the sample space S. Then the arithmetic operations create new variable:

$aX + bY$, $a, b \in \mathbb{R}$, $X \cdot Y$, X/Y, ...

Independence of the random variables X and Y

Let X and Y be two random variables with the following distributions:
$p_i = \mathrm{P}(X = x_i)$, $i = 1, 2, 3, \dots, n$,

$q_j = \mathrm{P}(Y = y_j)$, $j = 1, 2, 3, \dots, m$.

If

$$\mathrm{P}(X = x_i \text{ and } Y = y_j) = p_i q_j,\ i = 1, 2, 3, \dots, n,\ j = 1, 2, 3, \dots, m,$$

then the random variable X and Y are called the **independent variables.**

Comments

It follows from the condition above that if X and Y are independent then

$$\mathrm{P}(X \in A \text{ and } Y \in B) = \mathrm{P}(X \in A) \cdot \mathrm{P}(Y \in B)$$

for any $A, B \subseteq \mathbb{R}$.

Example. The random variable with an infinite expected value. Let X be a random variable having the following distribution

$$\Pr(X = k) = \frac{C}{k^2}, \quad k = 1, 2, 3, \ldots$$

where the constant C is defined as

$$C = 1 / \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Then the expected value

$$\mathrm{E}X = C \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Important discrete distributions

1. Two point discrete distribution: $X : S \to \{0, 1\}$, $\Pr(X = 0) = 1 - p$, $\Pr(X = 1) = p$, gdzie $0 < p < 1$,
   $\mathrm{E}X = p$, $\mathrm{D}^2[X] = pq$
   usually the event $\{X = 1\}$ is called "success" and the event $\{X = 0\}$ is called "failure".

2. Geometrical distribution with success probability p: $X : S \to \{0, 1, 2, 3, \ldots, n, \ldots\}$, $\Pr(X = k) = p(1-p)^{k-1}$, gdzie $0 < p < 1, k = 1, 2, 3, \ldots$, $\mathrm{E}X = \frac{1}{p}$, $\mathrm{D}^2[X] = \frac{1-p}{p^2}$

   Example. Suppose you toss a biased coin, with probability p for "head" and $q = 1 - p$ for "tail" repeatedly until a head turns up. Let X denote the number of tosses it takes until this happens, so that $\{X = k\}$ means $k - 1$ tails before the first head. Since then the favorable outcome is just the specific sequence $(\underbrace{T, T, T, \ldots, T}_{k-1}, H)$, its probability is

   $$p_k = \mathrm{P}(X = k) = q^{k-1}p, \; k = 1, 2, 3, 4, \ldots$$

   The random variable X is called the waiting time, for heads to fall, or more generally for a "success."

3. Binomial distribution (the Bernoulli scheme): $X : S \to \{0, 1, 2, 3, \ldots, n\}$, $\Pr(X = k) = \binom{n}{k} p^k q^{n-k}$, gdzie $0 < p < 1, k = 0, 1, 2, \ldots, n$, $q = 1 - p$, notation: $b(n, k, p) = \binom{n}{k} p^k q^{n-k}$, $\mathrm{E}X = np$, $\mathrm{D}^2[X] = npq$

Theorem 1. *Let $X_1, X_2, \ldots, X_n : S \to \{0,1\}$ be independent random variables having the same two-point distribution $\Pr(X_i = 0) = 1 - p$, $\Pr(X_i = 1) = p$, gdzie $0 < p < 1$, $i = 1, 2, \ldots, n$. Then the random variable $X = X_1 + X_2 + \cdots + X_n$ has the binomial distribution.*

Example. A die is rolled repeatedly n times. Let S_n denote the number of "sixth's" obtained. In our notation $S_n = X_1 + X_2 + \cdots + X_n$, where $X_i = 1$ if the "sixth" ocurred in the i-th roll, and $X_i = 0$ otherwise. Our probabilistic model is the Bernoulli scheme with the probability of "succes" $p = 1/6$. We find

$$p_k = \mathrm{P}(S_n = k) = \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k}, \quad k = 0, 1, 2, 3, 4, \ldots$$

4. The Poisson distribution with a parameter $\lambda > 0$ of the random variable $X : \Omega \to \{0, 1, 2, 3, \ldots, n, \ldots\}$:

   $\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, where $0 < p < 1,\ k = 0, 1, 2, 3, \ldots,$

   $\mathrm{E}X = \lambda$, $\mathrm{D}^2[X] = \lambda$