Problem set 04 – Solutions

Exercise 1.

P(A): Car

P(B): Imported

	Cars (A)	Light Trucks	**Total**
Domestic	23%	55%	78%
Imported (B)	8%	14%	22%
Total	31%	69%	**100%**

a) The vehicle is a light truck

$P(A^C) = 1 - P(A) = 0.69$

b) The vehicle is an imported car

P(A and B) or $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$P(A \cap B) = 0.31 + 0.22 - [\, 1 - P(A^c \cup B^c)]$

$P(A \cap B) = 0.31 + 0.22 - 0.45$

$P(A \cap B) = 0.08$

Exercise 2.

$S = \{(k_1, k_2) : k_1, k_2 \in \{1, 2, 3, 4, 5, 6\}$

$\mathbf{A} = \{(k_1, k_2) : k_1 + k_2 \text{ is odd}, k_1, k_2 \in \{1, 2, 3, 4, 5, 6\})\}$

or $\mathbf{A} = \{\text{even, odd}\} \cup \{(\text{odd, even}\}$

$\mathbf{B} = \{(k_1, k_2) : k_1 = 1 \text{ or } k_2 = 1\}$

or $\mathbf{B} = \{(1,k) : k \in \{1, 2, 3, 4, 5, 6\} \cup \{(k,1) : k \in \{1, 2, 3, 4, 5, 6\}$

$\mathbf{A} \cup \mathbf{B} = \{(k_1, k_2) : (k_1 + k_2 \text{ is odd}) \text{ or } (k_1 = 1 \text{ or } k_2 = 1)\}$

$\mathbf{A} \setminus \mathbf{B} = \{(k_1, k_2) : (k_1 + k_2 \text{ is odd}) \text{ and } (k_1 \neq 1 \text{ and } k_2 \neq 1\}$

$\mathbf{A} \cap \mathbf{B} = \{(1,k) : 1+k \text{ is odd}\} \cup \{(k,1) : k + 1 \text{ is odd}\}$

or $\mathbf{A} \cap \mathbf{B} = \{(1, 2), (1, 4), (1, 6)\} \cup \{(2, 1), (4, 1), (6, 1)\}$

$\mathbf{B} \setminus \mathbf{A} = \{(1,k) : 1+k \text{ is even}\} \cup \{(k,1) : k + 1 \text{ is even}\}$

or $\mathbf{B} \setminus \mathbf{A} = \{(1, 1), (1, 3), (1, 5)\} \cup \{(3, 1), (5, 1)\}$

Exercise 3.

$S = \{(t_1, t_2 \ldots\ldots t_n) : n \geq 2, t_i \in \{H, T\}, t_{n-1} = t_n, \text{ and } t_i \neq t_{i+1}\}$

a) $\mathbf{A} = \{(t_1, t_2 \ldots\ldots t_n) : 2 \leq n \leq 4, t_i \in \{H, T\}, t_{n-1} = t_n, \text{ and } t_i \neq t_{i+1}\}$
or $\mathbf{A} = \{HH, TT, HTT, THH, HTHH, THTT\}$

b) $\mathbf{B} = \{(t_1, t_2 \ldots\ldots t_n) : n \text{ is even}, t_i \in \{H, T\}, t_{n-1} = t_n, \text{ and } t_i \neq t_{i+1}\}$

c) **C** = {HTHTHTHT..., THTHTHTHTHTHT...}
Infinite alternation sequence

Exercise 4.

$S = \{(t_1, t_2 \ldots\ldots t_n) : n \geq 2, t_i \in \{H, T\}, t_{n-1} = t_n, \text{ and } t_i \neq t_{i+1}\}$

<u>Let's assign A as the result of the coin toss</u>

A = {H, T}, P(A) = ½

<u>For n times, two consecutive tosses, and n ≥ 2:</u>

$P(N = n)\ 2 \times (½)^n = \mathbf{2^{1-n}}, \quad n \geq 2$

a) $P(N \leq 6) = \sum_{n=2}^{6} 2^{1-n} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5}$

$P(N \leq 6) = 0.5 + 0.25 + 0.125 + 0.0625 + 0.03125$

$\mathbf{P(N \leqslant 6) = 0.96875}$

b) $P(N \text{ even}) = \sum_{m=1}^{\infty} P(N = 2m)$

$P(N = 2m) = 2^{1-2m} = 2 \times 2^{-2m} = 2 \times (¼)^m$

$P(N \text{ even}) = 2 \sum_{m=1}^{\infty} (\frac{1}{4})^m = 2 \times \frac{\frac{1}{4}}{1-\frac{1}{4}} = 2 \times ⅓ = ⅔$

P(N even) = 0.667

Solved differently:

ω = {HH, TT, HT, TH], for n=2

Two outcomes:

P(end) = ½, P(continue) = ½

But if it is odd, it continues and it resets. So,

P(even) = ½ + ½P(odd), P(odd) = 1- P(even)

P(even) = ½ + ½(1 - P(even))

P(even) + ½P(even) = ½ + ½ ⇔ 3/2P(even) = 1 ⇔ P(even) = ⅔

P(even) = 0.667

Exercise 5.

a) P(O) = 1 – [P(A) + P(B) + P(AB)]
P(O) = 1 – (0.40 + 0.11 + 0.04)
P(O) = 0.45

Blood type	**A**	**B**	**AB**	**O**
U.S. Probability	0.40	0.11	0.04	**0.45**

b) P(M) = P(B) + P(O)
P(M) = 0.11 + 0.45
P(M) = 0.56

Exercise 6.

a) P (US.O. ∩ C.O.) = P(US.O.) x P(C.O.)
P(US.O. ∩ C.O.) = 0.45 x 0.35
P(US.O. ∩ C.O.) = 0.1575

Blood type	**A**	**B**	**AB**	**O**
China Probability	0.27	0.26	0.12	0.35

b) $\sum P_{US}(t) x P_{Ch}(t),\ t \in \{A, B, AB, O\}$

- P_{US} (A) x P_{Ch} (A) = 0.40 x 0.27 = **0.108**
- P_{US} (B) x P_{Ch} (B) = 0.11 x 0.26 = **0.0286**
- P_{US} (AB) x P_{Ch} (AB) = 0.04 x 0.12 = **0.0048**
- P_{US} (O) x P_{Ch} (O) = **0.1575,** *from part a)*

P(s.b.t.) = 0.2989

Exercise 7.

- P(A+) = 0.40 x 0.84 = **0.336**
- P(A-) = 0.40 x 0.16 = **0.064**
- P(B+) = 0.11 x 0.84 = **0.0924**
- P(B-) = 0.11 x 0.16 = **0.0176**
- P(AB+) = 0.04 x 0.84 = **0.0336**
- P(AB-) = 0.04 x 0.16 = **0.0064**
- P(O+) = 0.45 x 0.84 = **0.378**
- P(O-) = 0.45 x 0.16 = **0.072**

Note: As a double check, you can sum up all the numbers. They should account for ***1****.*

Exercise 8.

a) **Not legitimate.**
The sum accounts for 2 and not 1
b) **Legitimate**
Even if it's not a standard deck, the sum accounts for 1
c) **Legitimate**

Exercise 9.

Language	English	French	Asian/Pacific	Other
Probability	**0.59**	0.23	0.07	0.11

a) P(E) = 1 – [P(F) + P(AP) + P(O)
P(E) = 1 – [0.23 + 0.07 + 0.11]
P(E) = 0.59

b) $P(E^C)$ = 1 - P(E)
$P(E^C)$ = 0.41

Exercise 10.

Topic	Adult	Financial	Health	Leisure	Products	Scams
Probability	0.145	0.162	0.073	0.078	0.210	0.142

a) P(O) = 1 – [P(A) + P(F) + P(H) + P(L) + P(P) + P(S)]
P(O) = 1 – (0.145 + 0.162 + 0.073 + 0.078 + 0.210 + 0.142)
P(O) = 1 – 0.810 ⇔ **P(O) = 0.190**

b) P(C) = P(A) + P(S)
P(C) = 0.145 + 0.142
P(C) = 0.287

Exercise 11.

	Hispanic (A)	Not Hispanic
Asian	0.000	0.036
Black	0.003	0.121
White (B)	0.060	0.691
Other	0.062	0.027

a) Yes, all probabilities sum up to 1
b) P(A) = 0.003 + 0.060 + 0.062 ⇔ **P(A) = 0.125**
c) $P(B^C)$ describes everyone who is not white
$P(B^C)$ = 1 – P(B) = 1 – (0.036 + 0.003 + 0.121 + 0.062 +0.027)
$P(B^C)$ = 0.249
d) $P(A^C \cap B)$. *From the table above,* **P = 0.691**

Exercise 12.

$P(A\cap B) = P(A) x P(B) = [(0.003 + 0.060 + 0.062) x (0.060 + 0.691)]$

$P(A\cap B) = 0.125 x 0.751$

$P(A\cap B) = 0.093875 \neq 0.060$ So, not independent

Exercise 13.

a) <u>For 456</u> (3 different numbers)
For the first number (e.g., 4), we have <u>**3**</u> positions
For the second number, we have <u>**2**</u> positions
And <u>**1**</u> position for the last one
So, distinct orders are 3 x 2 x 1 = **6**
P_{456}(win) = 6/1000 = 0.006

b) <u>For 212</u> (2 different numbers)
For the single number (1), I have <u>**3**</u> positions
1 option for the other two
So, distinct orders are 3 x 1 = **3**
P_{212}(win) = 3/1000 = 0.003

Exercise 14.

1. <u>$P(A\cup B) = P(A) + P(B) – P(A\cap B)$</u>
There are a couple of ways to prove this. Here is one simple way:

A∪B can be written as a union of two disjoint events:

- All outcomes of A
- All outcomes of B that are not in A (B\A)

So, in probabilistic terms:
$P(A\cup B) = P(A) + P(B\setminus A)$ *because they are disjoint* **(1)**

Similarly, B is also a sum of two disjoint events

- All outcomes that are in A **and** B
- All outcomes that are in B but not in A

$P(B) = P(A\cap B) + P(B\setminus A) \Leftrightarrow P(B\setminus A) = P(B) - P(A\cap B)$ **(2)**

Combine equations (1) and (2):
<u>**$P(A\cup B) = P(A) + P(B) – P(A\cap B)$**</u>

2. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$ **(3)**

From part a), we showed that:
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ **(4)**

And:
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

However, since $(A \cap C)$ and $(B \cap C)$ **overlap**, by applying the equation from part a):
$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$ **(5)**

Substituting equations (4) and (5) into (3):
$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) + P(C) - [(P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)]$

$$\mathbf{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

3. The above formula can be generalized by writing a pattern observed in parts 1. And 2.:

- Adding all single-event probabilities
- Subtracting the intersections of even-number events
- Adding the intersections of odd-number events

Thus,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(A_i \cap A_j) + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} P(A_i \cap A_j \cap A_k) - \ldots$$

until $+ \mathbf{(-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)}$