Problem set 05 – Solutions

Exercise 1.

<u>Binomial probability</u>

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n = 5$, $p = 0.05$

$P(\text{reject}) = P(X \geq 1) = 1 - P(X = 0)$

$$P(X = 0) = \binom{5}{0} p^0 (1 - 0.05)^{5-0}$$

$P(X = 0) = 1 \text{ x } 1 \text{ x } 0.95^5 \Leftrightarrow$ **$P(X = 0) = 0.7738$**

$P(\text{reject}) = 0.2262$

Exercise 2.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$p = 1/3$, $n = 5$, $k = 4$

$$P(X = 4) = \binom{5}{4} \left(\frac{1}{3}\right)^4 \left(1 - \frac{1}{3}\right)^{5-4}$$

$P(X = 4) = 5 \text{ x } 0.0123 \text{ x } 0.6667$

$P(X = 4) = 0.041$

Exercise 3

We need $P(x + y + z > 1)$

$P(x + y + z > 1) = 1 - P(x + y + z \leq 1)$

Imagine we have a <u>tetrahedron</u> (triangular pyramid) with points

(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) that encompass <u>the probabilistic area of</u>

<u>$P(x + y + z \leq 1)$</u>

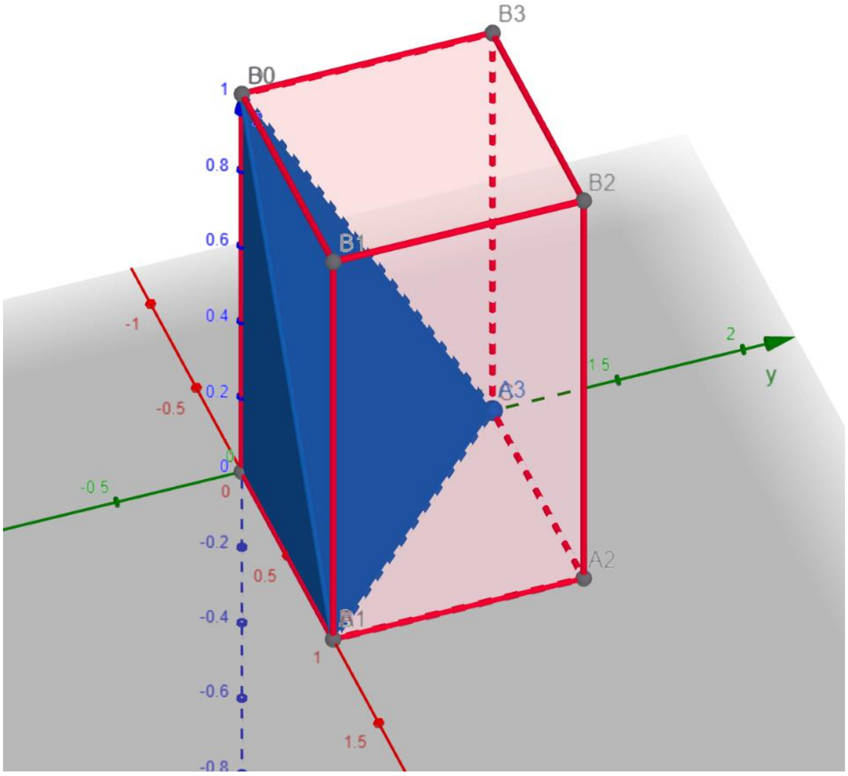
The illustration shows the probabilistic area in **blue** inside the interval [0,1] in **red**

We need to find the volume, V:

$V = ⅓ \; x \; B \; x \; h$. $B = ½ \; x$ base triangle x height triangle

$P(x + y + z > 1) = 1 - P(x + y + z \leq 1) = 1 - ⅓ \text{ x } ½ \text{ x } 1$

$\mathbf{P(x + y + z > 1) = \frac{5}{6} = 0.833}$



Exercise 4

$S = \{(k_1, k_2) : k_1, k_2 \in \{1, 2, 3, 4, 5, 6\}$

$A = \{(k_1, k_2)\} : k_1 + k_2 \text{ is odd}, k_1, k_2 \in \{1, 2, 3, 4, 5, 6\})\}$

or $A = \{\text{even, odd}\} \cup \{(\text{odd, even}\}$

$B = \{(k_1, k_2) : k_1 = 1 \text{ or } k_2 = 1\}$

or $B = \{(1,k) : k \in \{1, 2, 3, 4, 5, 6\} \cup \{(k,1) : k \in \{1, 2, 3, 4, 5, 6\}$

$A \cup B = \{(k_1, k_2) : (k_1 + k_2 \text{ is odd}) \text{ or } (k_1 = 1 \text{ or } k_2 = 1)\}$

$A \setminus B = \{(k_1, k_2) : (k_1 + k_2 \text{ is odd}) \text{ and } (k_1 \neq 1 \text{ and } k_2 \neq 1\}$

$A \cap B = \{(1,k) : 1{+}k \text{ is odd}\} \cup \{(k,1) : k + 1 \text{ is odd}\}$

or $A \cap B = \{(1, 2), (1, 4), (1, 6)\} \cup \{(2, 1), (4, 1), (6, 1)\}$

$B \setminus A = \{(1,k) : 1{+}k \text{ is even}\} \cup \{(k,1) : k + 1 \text{ is even}\}$

or $B \setminus A = \{(1, 1), (1, 3), (1, 5)\} \cup \{(3, 1), (5, 1)\}$

Exercise 5

P(BB) = α, P(GG) = β, P(mixed) = 1 − α − β

For twins, the firstborn to be a girl: P(first G) = ½

P(GB) = P(BG) = ½(1 − α − β)

We need <u>P(second G | first G)</u>

$P(\text{second G} \mid \text{first G}) = \frac{P(\text{second G and first G})}{P(\text{first G})}$

$P(\text{second G} \mid \text{first G}) = \frac{P(GG)}{P(GB)+P(GG)} = \frac{\beta}{\frac{1}{2}(1-\alpha-\beta)+\beta}$

$\mathbf{P(\text{second G} \mid \text{first G}) = \frac{2\beta}{1-a+\beta}}$

Exercise 6

<u>Hits:</u> $P(S_1H)$ = ½, $P(S_2H)$ = ⅔ , $P(S_3H)$ = ¾

<u>Misses:</u> $P(S_1M)$ = 1 − ½ = ½ , $P(S_2M)$ =1 − ⅔ = ⅓ $P(S_3M)$ = 1 − ¾ = ¼

We want $\mathbf{P(S_nM \mid 2H) = \frac{P(S_nM \text{ and } 2H)}{P(2H)}}$, for n = 1,2,3

$P(S_1M \text{ and } 2H) = (M, H, H) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \Leftrightarrow P(S_1M \text{ and } 2H) = \frac{6}{24}$

$P(S_2M \text{ and } 2H) = (H, M, H) = \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} \Leftrightarrow P(S_1M \text{ and } 2H) = \frac{3}{24}$

$P(S_3M \text{ and } 2H) = (H, H, M) = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} \Leftrightarrow P(S_1M \text{ and } 2H) = \frac{2}{24}$

$P(2H) = \frac{6}{24} + \frac{3}{24} + \frac{2}{24} \Leftrightarrow \mathbf{P(2H) = \frac{11}{24}}$

- **For n = 1,** $\mathbf{P(S_1M \mid 2H) = \frac{P(S_1M \text{ and } 2H)}{2H} = \frac{\frac{6}{24}}{\frac{11}{24}} \Leftrightarrow P(S_1M \mid 2H) = \frac{6}{11}}$
- **For n = 2,** $\mathbf{P(S_2M \mid 2H) = \frac{P(S_2M \text{ and } 2H)}{2H} = \frac{\frac{3}{24}}{\frac{11}{24}} \Leftrightarrow P(S_2M \mid 2H) = \frac{3}{11}}$
- **For n = 3,** $\mathbf{P(S_3M \mid 2H) = \frac{P(S_3M \text{ and } 2H)}{2H} = \frac{\frac{2}{24}}{\frac{11}{24}} \Leftrightarrow P(S_3M \mid 2H) = \frac{2}{11}}$

Exercise 7

Airline misses: $P(A_1) = 0.4$, $P(A_2) = 0.2$, $P(A_3) = 0.1$

We want $\mathbf{P(R_1| \text{ no arrival}) = \frac{P(R_1 \text{ and no arrival})}{P(\text{no arrival})}}$

$P(\text{no arrival}) = P(R_1) + P(R_2) + P(R_3)$

$P(R_1) = P(A_1) \Leftrightarrow P(R_1) = 0.4$

$P(R_2) = P(A_1^C \cap A_2) = [1 - P(A_1)] \text{ x } P(A_2) = 0.6 \text{ x } 0.2 \Leftrightarrow P(R_2) = 0.12$

$P(R_3) = P(A_1^C \cap A_2^C \cap A_3) = [1 - P(A_1)] \text{ x } [1 - P(A_2)] \text{ x } P(A_3)$

$P(R_3) = 0.6 \text{ x } 0.8 \text{ x } 0.1 \Leftrightarrow P(R_3) = 0.048$

$P(\text{no arrival}) = P(R_1) + P(R_2) + P(R_3) \Leftrightarrow \mathbf{P(\text{no arrival}) = 0.568}$

$P(R_1| \text{ no arrival}) = \frac{0.4}{0.568} \Leftrightarrow \mathbf{P(R_1| \text{ no arrival}) = 0.704}$

Exercise 8

P(A): Car

P(B): Imported

	Cars (A)	Light Trucks	**Total**
Domestic	23%	55%	78%
Imported (B)	8%	14%	22%
Total	31%	69%	**100%**

a) The vehicle is a light truck

$\mathbf{P(A^C) = 1 - P(A) = 0.69}$

b) The vehicle is an imported car

$P(A \text{ and } B)$ or $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$P(A \cap B) = 0.31 + 0.22 - [\ 1 - P(A^c \cup B^c)]$

$P(A \cap B) = 0.31 + 0.22 - 0.45$

$\mathbf{P(A \cap B) = 0.08}$

Exercise 9

A = {R = 6, W = 1} (single outcome (6, 1))

B = {R + W is even} (18 different outcomes)

We want: $\mathbf{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$

But, $A \subseteq B^c$

$\mathbf{P(A \mid B) = 0}$

or

$A \cap B = \{\varnothing\}$ therefore: $\mathbf{P(A \mid B) = 0}$

Exercise 10

If $A = \{x^2 + y^2 < 1\}$,

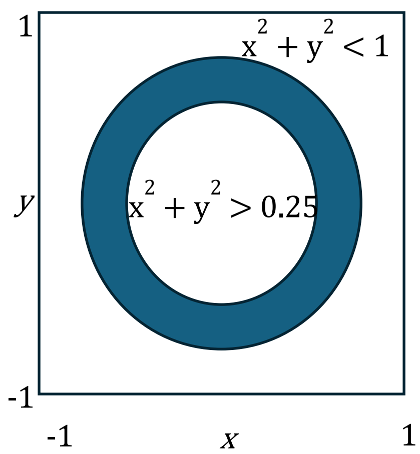
$B = \{x^2 + y^2 > 0.25\}$ and

$S = \{(x, y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$

We want $\mathbf{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$

If $x^2 + y^2 = r^2$

Imagine we have a <u>ring</u> with radius $0.5 < r < 1$ inside a square.



$A \cap B = \pi r_A{}^2 - \pi r_B{}^2 = \pi 1^2 - \pi 0.5^2 \Leftrightarrow A \cap B = \frac{3\pi}{4}$

$B = \alpha^2 - \pi r_B{}^2 = 2^2 - \pi 0.5^2 \Leftrightarrow B = \frac{16 - \pi}{4}$

$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3\pi}{4}}{\frac{16-\pi}{4}} = \frac{3\pi}{16 - \pi}$

$\mathbf{P(A \mid B) = 0.733}$

<u>CAREFUL!!!</u>

$$P(A \cap B) = \frac{\frac{3\pi}{4}}{P(S) = 4}$$