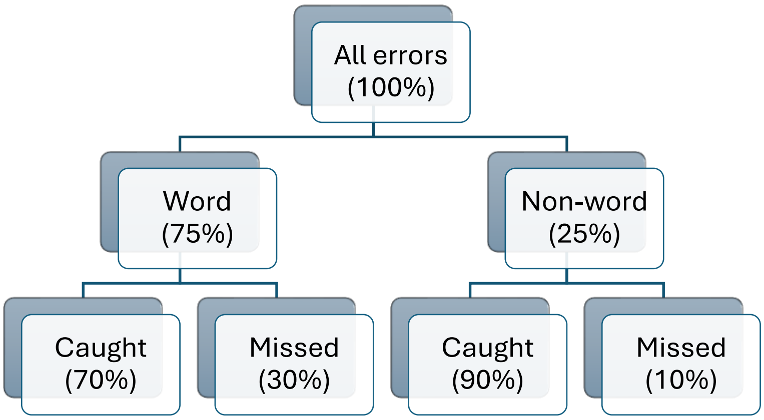
Problem set 06 – Solutions

Exercise 1.



$P(C) = P(C \mid NW)P(NW) + P(C \mid W)P(W)$

$P(C) = 0.70 \text{ x } 0.75 + 0.90 \text{ x } 0.25$

$\mathbf{P(C) = 0.75}$

Exercise 2.

<u>Binomial probability</u>

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

$n = 4,\ p = 0.5$

$P(X = k) = \binom{4}{k} \frac{1}{2}^k \left(1 - \frac{1}{2}\right)^{4-k}$

$\mathbf{P(X = k) = \binom{4}{k} \frac{1}{16}}$

1. <u>For $k = 0, 1, 2, 3, 4$</u>

$P(X = 0) = \binom{4}{0} \frac{1}{16} \Leftrightarrow \mathbf{P(X = 0) = \frac{1}{16}}$

$P(X = 1) = \binom{4}{1} \frac{1}{16} \Leftrightarrow \mathbf{P(X = 1) = \frac{4}{16}}$

$P(X = 2) = \binom{4}{2} \frac{1}{16} \Leftrightarrow \mathbf{P(X = 2) = \frac{6}{16}}$

$P(X = 3) = \binom{4}{3}\frac{1}{16} \Leftrightarrow \mathbf{P(X = 3) = \frac{4}{16}}$

$P(X = 4) = \binom{4}{4}\frac{1}{16} \Leftrightarrow \mathbf{P(X = 4) = \frac{1}{16}}$

2. <u>Events A, B, C</u>

A: four heads (calculated from part 1.)

$\mathbf{P(A) = P(X = 4) = \frac{1}{16}}$

B: Even number of heads

$P(B) = P(X = 0) + P(X = 2) + \mathbf{P(X = 4) = \frac{1}{16} + \frac{6}{16} + \frac{1}{16} = \frac{8}{16} = \frac{1}{2}}$

C: More heads than tails

$P(C) = P(X = 3) + P(X = 4) = \frac{4}{16} + \frac{1}{16} \Leftrightarrow \mathbf{P(C) = \frac{5}{16}}$

We see that <u>A ⊂ B</u> and <u>A ⊂ C</u>

$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{1}{16}}{\frac{1}{2}} \Leftrightarrow \mathbf{P(A \mid B) = \frac{1}{8}}$

$\mathbf{P(B \mid A) = 1}$ *since* $A \subset B$

If need to calculate:

$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{16}}{\frac{1}{16}} \Leftrightarrow \mathbf{P(B \mid A) = 1}$

$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} = \frac{\frac{1}{16}}{\frac{5}{16}} \Leftrightarrow \mathbf{P(A \mid C) = \frac{1}{5}}$

$\mathbf{P(C \mid A) = 1}$ *since* $A \subset C$

$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{1}{16}}{\frac{5}{16}} \Leftrightarrow \mathbf{P(B \mid C) = \frac{1}{5}}$

$P(C \mid B) = \frac{P(B \cap C)}{P(B)} = \frac{\frac{1}{16}}{\frac{1}{2}} \Leftrightarrow \mathbf{P(B \mid C) = \frac{1}{8}}$

Exercise 3.

We want **P(G)**

P(G) = 1 – P(D)

P(G) = 1 – [P(W.I.)P(D.I.) + P(W.II.)P(D.II.) + P(W.III.)P(D.III.)]

P(G) = 1 – [0.30 x 0.02 + 0.45 x 0.01 + [1 – (0.30 + 0.45)] x 0.03]

P(G) = 0.982

Exercise 4.

P(B) = P(G) = 0.5

We need to find out **P(BGG | B1)**

$P(BGG \mid B1) = \frac{P(B1 \cap G2 \cap G3)}{P(B1)} = \frac{0.5 \, x \, 0.5 \, x \, 0.5}{0.5}$

P(BGG | B1) = 0.25

Can it be solved as a binomial probability?

A very situational example, but yes

If we take G: success **and**

We lock as a certainty that the first child is a boy

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; p = 0.5, \; n = 2, \; k = 2$

$P(X = 2) = \binom{2}{2} 0.5^2 (1 - 0.5)^{2-2}$

$\boldsymbol{P(X = 2) = 0.25}$

Given the circumstances, it can be solved as binomial but always better to go with the conditional probabilities or Bayes theorem.

We need to find out P(two girls | one boy selected randomly)

S = {2^n, so three children, 2^3} (8 total outcomes)

A = {two girls in a family of three} = {BGG, GBG, BGG}

C = {randomly selected child is a boy}

$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$

$P(A \cap C) = P(C \mid A)P(A) = \frac{1}{3} x \frac{3}{8} = \frac{1}{8}$

$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{8}}{\frac{1}{2}}$

P(two girls | one boy selected randomly) = 0.25

Exercise 5.

$P(A) = P(B) = 0.5$

$P(D_A) = 0.05$, $P(D_B) = 0.01$

We need to find $\mathbf{P(G_2 \mid G_1)}$

$P(G_1) = P(A)P(G_A) + P(B)P(G_B)$

$P(G_1) = P(A)[1 - P(D_A)] + P(B)[1 - P(D_B)] = 0.5 \text{ x } 0.95 + 0.5 \text{ x } 0.99$

$\mathbf{P(G_1) = 0.97}$

$P(G_1 \cap G_2) = P(A)P(G_A)^2 + P(B)P(G_B)^2 = 0.5 \text{ x } 0.95^2 + 0.5 \text{ x } 0.99^2$

$\mathbf{P(G_1 \cap G_2) = 0.941}$

$P(G_2 \mid G_1) = \frac{P(G1 \cap G2)}{P(G1)} = \frac{0.941}{0.97} \Leftrightarrow$ $\mathbf{P(G_2 \mid G_1) = 0.9704}$

Exercise 6.

Johnny wins if **HHT**, Greg wins if **THT**

How it was solved in the lecture

From Start (p_S), we can get only two outcomes, H or T, with equal probability.

$p_S = ½p_H + ½p_T$

Then we examine the different sequences after the first toss. We check first for H

For p_H, again, the same two outcomes with equal probability

$p_H = ½p_{HH} + ½p_{HT}$

But, while HH is a wanted outcome (HHT), HT is **not**, so it resets to T (because of THT) and thus, $\mathbf{p_{HT} = p_T}$

So, $p_H = ½p_{HH} + ½p_T$

For p_{HH}, in the same context:

$p_{HH} = ½p_{HHH} + ½p_{HHT}$

But, HHH resets to HH (not a desirable outcome) so $\mathbf{p_{HHH} = p_{HH}}$

And with HHT, Johnny wins so $\mathbf{p_{HHT} = 1}$

$p_{HH} = ½p_{HHH} + ½p_{HHT}$

$p_{HH} = ½p_{HH} + ½ \text{ x } 1 \Leftrightarrow \mathbf{p_{HH} = 1}$

If the first outcome from the Start, p_S was T (p_T)

$p_T = ½p_{TH} + ½p_{TT}$ but, TT is not a wanted outcome so resets to T and thus, $\mathbf{p_{TT} = p_T}$

$p_T = ½p_{TH} + ½p_T$ and by rearranging:

$\mathbf{p_T = p_{TH}}$

For p_{TH}, again two possible outcomes: THH and THT.

However, $\mathbf{p_{HHT} = 0}$ as Johnny loses and THH resets to HH and we determined above $\mathbf{p_{HH} = 1 = p_{THH}}$. So:

$p_{TH} = ½p_{THH} + ½p_{THT} = ½ \text{ x } 1 + ½ \text{ x } 0$

$\mathbf{p_{TH} = ½}$

And since, $\mathbf{p_T = p_{TH}}$ (determined above) → $\mathbf{p_T = ½}$

Back to the Start (p_S)

$p_S = ½p_H + ½p_T$ and $p_H = ½p_{HH} + ½p_T$

$p_S = ½ (½p_{HH} + ½p_T) + ½p_T$

$p_S = ¼p_{HH} + ¾p_T$ and $p_{HH} = 1$, $p_T = ½$

$p_S = ¼ \times 1 + ¾ \times ½$

$\mathbf{p_S = {}^{5}/_{8}}$

Therefore **P(J) =** $\mathbf{{}^{5}/_{8}}$ and

P(G) = P(J^C) = $\mathbf{{}^{3}/_{8}}$

With conditional probabilities

We track the last two flips, because due to the desired patterns, we investigate three outcomes of two flips: **HH, TH,** and **TT** (which will reset to T) (HT

P(Johnny | HH) = 1

P(Johnny | TH) = ½ as THH → HH → Johnny wins and if THT → Greg wins

P(Johnny | T) = ½ as TH → ½ (see above) and TT → resets

P(Johnny | first flip H) = $\mathbf{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}}$

P(Johnny | first flip T) = $\mathbf{\frac{1}{2}}$

P(J) = P(first flip H)P(Johnny | first flip H) + P(first flip T)P(Johnny | first flip T)

$P(J) = \frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{2}$

P(J) = $\mathbf{{}^{5}/_{8}}$

Exercise 7.

All hypotheses have equal probability

H_k: there are ***k*** white balls in the urn ($k = 0, 1, ..., n$)

$P(H_k) = \frac{1}{n+1}$, $k = 0, 1, ..., n$

After adding one white ball: $\boldsymbol{k+1}$ white balls and $\mathbf{n+1}$ total balls

So, $\mathbf{P(W \mid H_k) = \frac{k+1}{n+1}}$

We want **P(W)**

Total probability equation

$$P(W) = \sum_{k=0}^{n} P(W \mid H_k)P(H_k)$$

$$P(W) = \sum_{k=0}^{n} \left(\frac{k+1}{n+1} \times \frac{1}{n+1}\right)$$

$$P(W) = \left(\frac{1}{n+1}\right)^2 \sum_{k=0}^{n} (k+1)$$

$$P(W) = \left(\frac{1}{n+1}\right)^2 \frac{(n+1)(n+2)}{2}$$

$$\mathbf{P(W) = \frac{\boldsymbol{n}+2}{2(\boldsymbol{n}+1)}}$$

Exercise 8.

If we define

A: live to 60 → $P(A) = 0.89835$

B: live to 80 → $P(B) = 0.57062$

$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$, but $B \subset A$

$P(B \mid A) = \frac{P(B)}{P(A)} = \frac{0.57062}{0.89835}$

$\mathbf{P(B \mid A) = 0.6352}$

Exercise 9.

Binomial probability

$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

$n = 5,\ p = 0.5$

$P(X = k) = \binom{5}{k} \frac{1}{2}^k \left(1 - \frac{1}{2}\right)^{5-k}$

$\mathbf{P(X = k) = \binom{5}{k} \frac{1}{32}}$

For $k = 0, 1, 2, 3, 4, 5$

$P(X = 0) = \binom{5}{0} \frac{1}{32} \Leftrightarrow P(X = 0) = \frac{1}{32}$

$P(X = 1) = \binom{5}{1} \frac{1}{32} \Leftrightarrow P(X = 1) = \frac{5}{32}$

$P(X = 2) = \binom{5}{2} \frac{1}{32} \Leftrightarrow P(X = 2) = \frac{10}{32}$

$P(X = 3) = \binom{5}{3} \frac{1}{32} \Leftrightarrow P(X = 3) = \frac{10}{32}$

$P(X = 4) = \binom{5}{4} \frac{1}{32} \Leftrightarrow P(X = 4) = \frac{5}{32}$

$P(X = 5) = \binom{5}{5} \frac{1}{32} \Leftrightarrow P(X = 5) = \frac{1}{32}$

$P(X > 3) = P(X = 4) + P(X = 5)$

$P(X > 3) = \frac{5}{32} + \frac{1}{32} = \frac{6}{32}$

$\mathbf{P(X > 3) = 0.1875}$

Exercise 10.

We define the events

A: the coin is two-headed $\mathbf{P(A) = \frac{1}{100}}$ hypothesis

B: the coin is normal $\mathbf{P(B) = \frac{99}{100}}$ hypothesis

C: 6 heads in a row observation

We want **P(A | C)**

From <u>Bayes formula</u>: $P(A \mid C) = \frac{P(C \mid A)P(A)}{P(C \mid A)P(A) + P(C \mid B)P(B)}$

$\mathbf{P(C \mid A) = 1}$

$\mathbf{P(C \mid B) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}}$

$$P(A \mid C) = \frac{P(C \mid A)P(A)}{P(C \mid A)P(A) + P(C \mid B)P(B)}$$

$$P(A \mid C) = \frac{1 \text{ x } \frac{1}{100}}{1 \text{ x } \frac{1}{100} + \frac{1}{64} \text{ x } \frac{99}{100}} = \frac{\frac{1}{100}}{\frac{1}{100} + \frac{99}{6400}} = \frac{\frac{1}{100}}{\frac{163}{6400}}$$

$$\mathbf{P(A \mid C) = \frac{64}{163} = 0.393}$$